# The placeholder view of assumptions and the Curry–Howard correspondence (extended abstract)*

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**Abstract.** Proofs from assumptions are amongst the most fundamental reasoning techniques. Yet the precise nature of assumptions is still an open topic. One of the most prominent conceptions is the placeholder view of assumptions generally associated with natural deduction for intuitionistic propositional logic. It views assumptions essentially as holes in proofs (either to be filled with closed proofs of the corresponding propositions via substitution or withdrawn as a side effect of some rule), thus in effect making them an auxiliary notion subservient to proper propositions. The Curry-Howard correspondence is typically viewed as a formal counterpart of this conception. In this talk, based on my paper of the same name (*Synthese*, 2020), I will argue against this position and show that even though the Curry-Howard correspondence typically accommodates the placeholder view of assumptions, it is rather a matter of choice, not a necessity, and that another more assumption-friendly view can be adopted.

**Keywords:** placeholder view of assumptions · assumption withdrawing · Curry–Howard correspondence · natural deduction · intuitionistic propositional logic.

## 1 Introduction

Proofs from assumptions are amongst the most fundamental reasoning techniques. Yet the precise nature of assumptions is still an open topic. One of the most prominent conceptions is the placeholder view of assumptions generally associated with natural deduction for intuitionistic propositional logic. It views assumptions essentially as holes in proofs (either to be filled with closed proofs of the corresponding propositions via substitution or withdrawn as a side effect of some rule), thus in effect making them an auxiliary notion subservient to

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proper propositions (see, e.g., [15], p. 5). The Curry-Howard correspondence is typically viewed as a formal counterpart of this conception (recently, see, e.g., [13]). I this talk, based on my paper [8], I will argue against this position and show that even though the Curry-Howard correspondence typically accommodates the placeholder view of assumptions, it is rather a matter of choice, not a necessity, and that another more assumption-friendly view can be adopted.

*Assumption withdrawing.* The rule for implication introduction from natural deduction for intuitionistic propositional logic is arguably the best-known example of the assumption withdrawing rule:

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \supset B}$$

It prescribes the following inference step: if we can derive $B$ from assumption $A$, then we can derive $A \supset B$ and withdraw the initial assumption $A$ (it is worth noting that other assumptions than $A$ may be used in deriving $B$ and those remain open after discharging $A$). Note that this rule effectively embodies the deduction theorem from standard axiomatic systems. In other words, the implication introduction rule is internalizing structural information from the proof level ("$B$ is derivable from $A$") to the propositional level ("$A$ implies $B$").[1]

The problematic aspect of this and other assumption withdrawing rules stems from the fact that it behaves differently from the non-assumption withdrawing rules. More specifically, with the implication introduction rule we are deriving the proposition $A \supset B$ not from other propositions as with other standard rules (e.g., conjunction introduction), but from a hypothetical proof. To put it differently, the inference step validated by the implication introduction takes us from a *derivation* starting with a hypothesis to a proposition, not just from propositions to another proposition as do rules without assumptions.[2]

For example, consider the following simple proof of the theorem $A \supset ((A \supset B) \supset B)$ of propositional logic:

$$\dfrac{\dfrac{\dfrac{[A \supset B]^1 \qquad [A]^2}{B}\,{\supset}\mathrm{E}}{(A \supset B) \supset B}\,{\supset}\mathrm{I}_1}{A \supset ((A \supset B) \supset B)}\,{\supset}\mathrm{I}_2$$

[1] [13] describes this as a two-layer system. Note that, strictly speaking, the assumptions are not really withdrawn, they are rather incorporated into the propositional level in the form of an antecedent.

[2] This non-standard behaviour is also the reason why [10] describes assumption withdrawing rules as improper rules and introduces the distinction between inference rules and deductions rules. For more, see [10], [7].

We start by making two assumptions $A \supset B$ and $A$. Applying the implication elimination rule (modus ponens) we derive $B$. What follows are two consecutive applications of implication introduction rule, first withdrawing the assumption $A \supset B$, the second withdrawing the assumption $A$. Note that it is the fact that $B$ is derivable from $A \supset B$ together with $A$ that warrants the application of the implication introduction rule and the derivation of the corresponding proposition $(A \supset B) \supset B$, at that moment still depending on the assumption $A$. Analogously with the second application of the implication introduction rule that withdraws this remaining assumption.

A proof that relies on no assumptions is called a closed proof. If a proof depends on some assumptions that are yet to be withdrawn (i.e., open/active assumptions) it is called an open proof. For example, our derivation of $A \supset ((A \supset B) \supset B)$ constitutes a closed proof, since both assumption were withdrawn in the course of the derivation. Assuming we would not have carried out the last inference step, we would get an open proof:

$$\dfrac{\dfrac{[A \supset B]^1 \qquad A}{B}\supset\mathrm{E}}{(A \supset B) \supset B}\supset\mathrm{I}_1$$

since the assumption $A$, upon which the derivation of $((A \supset B) \supset B)$ depends, is still active.

Closed proofs are usually preferred to open ones for the simple reason that closed proofs are generally viewed as the fundamental notion in standard proof-theoretic systems. From this perspective, assumptions are just temporary holes in the proof that are preventing us from reaching a closed proof. These open holes can be are either completely discarded via assumption withdrawing rules or filled in with other already closed proofs via substitution. This is the reason why [13] and others[3] call this the placeholder view of assumptions: active assumptions are just auxiliary artefacts of the employed proof system that behave differently than proper propositions, i.e., propositions that do not appear as assumptions.

*The Curry-Howard correspondence.* The placeholder view of assumptions is also supported to a large extent by the Curry-Howard correspondence in its basic form which links typed lambda calculus and implicational fragment of intuitionistic propositional logic.[4] Under this correspondence, natural deduction assumptions correspond to free variables of lambda calculus, which fits well with the interpretation of assumptions as open holes in the proof.

For example, assuming only the implicational fragment of intuitionistic propositional natural deduction, we get the following correspondences between the propositional and functional dimensions of the Curry-Howard correspondence:

[3] See, e.g., [1]
[4] See, e.g., [14].

| Natural Deduction | Lambda Calculus |
|---|---|
| assumption | free variable |
| implication introduction | function abstraction |
| implication elimination | function application |

Under this correspondence, the implication introduction rule will then look as follows:

$$\frac{\begin{array}{c}[x:A]\\ \vdots \\ b(x):B\end{array}}{\lambda x.b(x):A\supset B}$$

Note that the act of withdrawing the assumption $A$ corresponds to $\lambda$-binding of the free variable $x$. The whole proof of the theorem $A \supset ((A \supset B) \supset B)$ would then proceed in the following way:

$$\cfrac{\cfrac{\cfrac{[x:A\supset B]^1 \qquad [y:A]^2}{xy:B}\supset\mathrm{E}}{\lambda x.xy:(A\supset B)\supset B}\supset\mathrm{I}_1}{\lambda y.\lambda x.xy:A\supset((A\supset B)\supset B)}\supset\mathrm{I}_2$$

with the concluding proof object (closed term) $\lambda y.\lambda x.xy$ with no free variables representing the final closed proof with no active assumptions. In contrast, the open proof discussed earlier:

$$\cfrac{\cfrac{[x:A\supset B]^1 \qquad y:A}{xy:B}\supset\mathrm{E}}{\lambda x.xy:(A\supset B)\supset B}\supset\mathrm{I}_1$$

concludes with the proof object $\lambda x.xy$ that still contains the free variable $y$ corresponding to the yet to be withdrawn assumption $A$.

*The placeholder view of assumptions and consequence statements.* The Curry-Howard correspondence is generally viewed as incorporating the placeholder view of assumptions. Probably most recently, this point was explicitly made in [13]. Furthermore, in the same paper Schroeder-Heister advocates for a more general concept of inference that takes us not from propositions to other propositions, but from (inferential) consequence statements $A \models B$ to other consequence statements in order to, amongst other things, equalize the status of assumptions and assertions.[5] The general form of inference rules he discusses is the following:

[5] Strictly speaking, we should be writing $A \models_{\mathbb{D}} B$, i.e., that $A \models B$ can be derived with respect to a set of definitional clauses $\mathbb{D}$ (see [12]), but for simplicity we omit these considerations.

$$\frac{A_1 \models B_1 \quad \dots \quad A_n \models B_n}{C \models D}$$

where the antecedents can be empty and its correctness means that whenever $A_1 \models B_1, \dots, A_n \models B_n$, then $C \models D$. As Schroeder-Heister explains:

> This corresponds to the idea that in natural deduction, derivations can depend on assumptions. Here this dependency is expressed by non-empty antecedents, as is the procedure of the sequent calculus. Our model of inference is the sequent-calculus model... ([12], p. 938)

To show that this rule is correct, we have demonstrated that given the grounds for the premises (denoted as $g : A \models B$) we can construct grounds for the conclusion. In other words, the grounds of the conclusion have to contain some operation $f$ transforming the grounds for the premises to the grounds for the conclusion. Schematically:

$$\frac{g_1 : A_1 \models B_1 \quad \dots \quad g_n : A_n \models B_n}{f(g_1, \dots g_n) : C \models D}$$

Schroeder-Heister comments on this rule as follows:

> ... [H]andling of grounds in the sense described is different from that of terms in the typed lambda calculus. When generating grounds from grounds according to [the rule immediately above], we consider grounds for whole sequents, whereas in the typed lambda calculus terms representing such grounds are handled within sequents. So the notation $g : A \models B$ we used above, which is understood as $g : (A \models B)$, differs from the lambda calculus notation $x : A \vdash t : B$, where $t$ represents a proof of $B$ from $A$ and the declaration $x : A$ on the left side represents the assumption $A$. ([12], p. 939)

However, it should be mentioned that he left it "open how to formalize grounds and their handling." (ibid., p. 938) I will argue that even though lambda calculus with the Curry-Howard interpretation can be seen as embodying the placeholder view of assumptions in the intuitionistic propositional logic, within the family of Curry-Howard correspondence based systems we can consider a generalized approach that is free of this view. This generalized approach will treat consequence statements $A \models B$ as higher-order functions $A \Rightarrow B$ that can be naturally captured in Martin-Löf's constructive type theory ([4]), specifically in its higher-order presentation (see [5], [6]).

*Function-based approach to assumptions.* Let us return to the implication introduction rule. Adopting the sequent-style notation for natural deduction,[6] we can rewrite this rule as follows:

$$\frac{x : A \vdash b(x) : B}{\vdash \lambda x.b(x) : A \supset B}$$

[6] See, e.g., Gentzen's system NLK, discussed in [9].

where the symbol $\vdash$ is used to separate assumptions from (derived) propositions.

Notice that the derivation of $B$ from $A$ is coded with an abstraction term from lambda calculus, which means it captures some sort of a function. Reasoning backwards, this should mean that between the assumption (context) and the conclusion (asserted proposition) has to be a relationship that can be understood functionally, otherwise, we would have nothing to code via lambda terms. To put it differently, there has to be some more fundamental notion of a function at play that we are coding through the concrete abstraction term.

We can try to capture this observation via the following rule:

$$\frac{x : A \vdash b(x) : B}{f : A \Rightarrow B}$$

where $f$ is to be understood as exemplifying the more fundamental notion of a function that takes us from $A$ to $B$.

Note that this rule can be roughly understood as the opposite of the implication introduction rule that goes in the other direction: while the implication introduction rule makes the hypothetical derivation "from $A$ is derivable $B$" in its premise more concrete in the form of implication proposition $A \supset B$ and the corresponding lambda term $\lambda x.b(x)$, this rule makes the derivation more general in the sense that it is now considered as a function $f$ (not specifically a lambda term) from $A$ to $B$. Also notice that assumptions are no longer placeholders or contexts, but types of arguments for the function $f$ capturing the corresponding derivation. In other words, assumptions now stand equal to proper propositions, they are not just an auxiliary notion captured via free variables.

Furthermore, capturing derivations in this way allows us to consider grounds for the whole consequence statements as Schroeder-Heister required, not just grounds for the conclusions under some assumptions. More specifically, treating consequence statement $A \models B$ as a function type $A \Rightarrow B$ (in accord with the Curry-Howard correspondence) and a ground $g$ as an object $f$ of this type, we can reformulate the general rule as follows (see [12], p. 938):

$$\frac{g_1 : A_1 \Rightarrow B_1 \quad \ldots \quad g_n : A_n \Rightarrow B_n}{f(g_1, \ldots g_n) : C \Rightarrow D}$$

*Formalization.* So far, I have treated $f : A \Rightarrow B$ informally to mean "$f$ is a function from $A$ to $B$". Utilizing Martin-Löf's constructive type theory ([4]), specifically its higher-order presentation ([5], [6]), we can capture it more rigorously as a higher-order judgment of the form $(x)b : (A)B$. To explain why, let us return to the hypothetical judgment $x : A \vdash b(x) : B$ that appears as the sole premise of the implication introduction rule. It tells us that we know $b(a)$ to be a proof of the proposition $B$ assuming we know $a$ to be a proof of the proposition $A$. In other words, the hypothetical judgment $x : A \vdash b(x) : B$ can be seen as stating that $b(x)$ is a function with domain $A$ and range $B$.[7] This fact, however, cannot be stated directly in the lower-order presentation of constructive

[7] See [4].

type theory. Thus we move towards the higher-order presentation, which is as a generalization of the lower-order presentation using a more primitive notion of a type. The higher-order variant of constructive type theory allows us to form a higher-order notion of a function which can be used to capture the function hidden behind the hypothetical judgment $x : A \vdash b(x) : B$ as an object $(x)b$ of type $(A)B$. Consequently, $(x)b : (A)B$ can then be used to interpret our statement $f : A \Rightarrow B$, as was required. In other words, $(x)b : (A)B$ can be understood as a higher-order judgment declaring that we have (potentially open) derivation of $B$ from $A$ captured by the function $(x)b$.

It is important to emphasize that the higher-order function type $(A)B$ cannot be conflated with the lower-order function type $A \supset B$. The most basic reason is that they are inhabited by different objects: the former by functions, the latter by elements specified by $\supset$-introduction rule, i.e., objects of the form $\lambda x.b(x)$ that are used to code functions. More generally, the notion of a function behind the type $A \supset B$ is parasitic on a more fundamental notion of a function behind the type $(A)B$.[8] From the logical point of view, the main reason we should avoid merging $(A)B$ and $A \supset B$ is that $A$ in $(A)B$ is an assumption of derivation, while $A$ in $A \supset B$ is an antecedent of implication, hence they are objects of different inferential roles. This is perhaps best illustrated by the fact that assuming some function $f$ of type $(A)B$ essentially corresponds to assuming a rule $\frac{\mathrm{A}}{\mathrm{B}}$ in Schroeder-Heister's natural deduction with higher-level rules ([11]).

*Conclusion.* In this talk, I have argued that the Curry-Howard correspondence is not necessarily connected with the placeholder view of assumptions generally associated with natural deduction systems for intuitionistic propositional logic. Although in the basic form of this correspondence, assumptions, which correspond to free variables, can indeed be thought of as just holes to be filled, we can consider also a functional approach where derivations from assumptions are regarded as functions (see [8]). On this account, assumptions are no longer just placeholders but domains of the corresponding functions. From the logical point of view, this move corresponds to the shift from reasoning with propositions to reasoning with consequence statements.

## References


1. Francez, N.: Proof-theoretic Semantics. College Publications (2015)
2. Klev, A.: A comparison of type theory with set theory. In: Centrone, S., Kant, D., Sarikaya, D. (eds.) Reflections on the Foundations of Mathematics. Springer, Cham (2019). https://doi.org/10.1007/978-3-030-15655-8_12
3. Klev, A.: Name of the Sinus Function. In: Sedlár, I., Blicha, M. (eds.) The Logica Yearbook 2018. College Publications, London (2019)
4. Martin-Löf, P.: Intuitionistic type theory: Notes by Giovanni Sambin of a series of lectures given in Padua, June 1980. Bibliopolis, Napoli (1984)


[8] See [2], [3].

5. Nordström, B., Petersson, K., Smith, J.M.: Programming in Martin-Löf's type theory: an introduction. International series of monographs on computer science, Clarendon Press (1990)
6. Nordström, B., Petersson, K., Smith, J.M.: Martin-Löf's type theory, Handbook of logic in computer science: Volume 5: Logic and algebraic methods. Oxford University Press, Oxford (2001)
7. Pezlar, I.: Towards a More General Concept of Inference. Logica Universalis **8**(1) (2014). https://doi.org/10.1007/s11787-014-0095-3
8. Pezlar, I.: The Placeholder View of Assumptions and the Curry–Howard Correspondence. Synthese (2020). https://doi.org/10.1007/s11229-020-02706-z
9. von Plato, J.: Gentzen's proof systems: byproducts in a work of genius. Bull. Symbolic Logic **18**(3), 313–367 (2012). https://doi.org/10.2178/bsl/1344861886
10. Prawitz, D.: Natural Deduction: A Proof-theoretical Study. Almqvist & Wiksell, Stockholm (1965)
11. Schroeder-Heister, P.: A natural extension of natural deduction. Journal of Symbolic Logic **49**(4), 1284–1300 (12 1984). https://doi.org/10.2307/2274279
12. Schroeder-Heister, P.: The categorical and the hypothetical: a critique of some fundamental assumptions of standard semantics. Synthese **187**(3), 925–942 (8 2012). https://doi.org/10.1007/s11229-011-9910-z
13. Schroeder-Heister, P.: Open Problems in Proof-Theoretic Semantics. In: Piecha, T., Schroeder-Heister, P. (eds.) Advances in Proof-Theoretic Semantics, pp. 253–283. Springer, Cham (2016). https://doi.org/10.1007/978-3-319-22686-6_16
14. Sørensen, M.H., Urzyczyn, P.: Lectures on the Curry-Howard Isomorphism, Volume 149 (Studies in Logic and the Foundations of Mathematics). Elsevier Science Inc., New York, NY, USA (2006)
15. Troelstra, A.S., Schwichtenberg, H.: Basic Proof Theory. Cambridge University Press, Cambridge (2000). https://doi.org/10.1017/cbo9781139168717