From explosion to implosion: a new justification for the ex falso quodlibet rule

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Abstract. In this paper, we consider the ex falso quodlibet rule (EFQ) as a derived rule and propose a new justification for it based on a rule we call the collapse rule. The collapse rule is a mix between EFQ and disjunctive syllogism (DS). Informally, it says that a choice between a proposition A and $\perp$, which is understood as nullary disjunction, is no choice at all and it defaults to A. Thus, we can regard it as capturing the idea of an implosion principle in contrast to EFQ's explosion principle. Furthermore, we show that the collapse rule can also be used to justify DS and that all these three rules have the same deductive strength: they are all interderivable. Thus, the discussions about the justification of EFQ or DS be reduced to a discussion about the justification of the collapse rule.

keywords: ex falso quodlibet, falsity rule, absurdity rule, principle of explosion, law of Pseudo-Scotus, disjunctive syllogism, natural deduction

1 Introduction

The ex falso quodlibet rule (EFQ; also known as the law of Pseudo-Scotus, or simply the falsity or absurdity rule):

$$\frac{\perp}{A}\ \text{EFQ}$$

directly capturing the principle of explosion, i.e., the idea that absurdity denoted by $\perp$ entails any proposition, is a precarious one.

Many non-classical logics, e.g., relevant logic and paraconsistent logic, reject it outright for various reasons but even those logics that accept it, most notably intuitionistic and classical logic, often have trouble with presenting good, philosophically satisfying reasons when pushed for an answer why they do so.[1] It will be the latter group of explosive logics, specifically intuitionistic and classical

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[1] Why indeed should anything follow from absurdity? It was not always the case, see, e.g., Hewitt (2022).

logic in the setting of natural deduction, that will be the main subject of this paper.[2]

The question of the validity or the legitimacy of the EFQ rule is tightly connected to the question of what kind of a rule it is as different kinds of rules have different means of justifications. For example, if we were to regard it as a logical/operational rule, then it should be either an introduction rule or an elimination rule. If it is an introduction rule, then it should be self-justifying; if it is an elimination rule, then it should be justifiable according to the corresponding introduction rule.[3] If we were to regard it as a structural rule, then some kind of a metalanguage justification should be provided.

Whether we view EFQ as a logical or a structural rule ultimately depends on the view towards the absurdity itself. If we consider it a logical constant/nullary logical connective, then EFQ has to be a logical rule. On the other hand, if we consider it just a marker for dead ends of derivations,[4] then EFQ has to be a structural rule.[5]

In this paper, we will be interested in the logical approach to EFQ and $\perp$, specifically, we will regard $\perp$ as a nullary disjunction.[6] We will provide a new way of justifying EFQ that to our best knowledge has not been considered so far in the literature. The main idea is that EFQ is a derived rule of inference, i.e., it is not a basic rule of the system in which it appears. This means that its justification depends on other rules of the system. Namely, it relies on the introduction rule for disjunction and a new rule we call the *collapse rule*:[7]

$$\frac{A \vee \perp}{A}\,\text{collapse}$$

which we will explain more in-depth later.

The corresponding derivation in a tree-like notation of natural deduction is very simple:

$$\cfrac{\cfrac{\perp}{A \vee \perp}\,\vee\mathrm{I}_r}{A}\,\text{collapse}$$

and it justifies the derived "short-cut" rule:

[2] As EFQ is utilized by both classical and intuitionistic logic, everything discussed in this paper is meant to apply to both of them, unless explicitly stated otherwise.

[3] We adopt here a view that privileges the introduction rules over the elimination rules by regarding them as the sole meaning-giving rules (also known as the verificationist view, see, e.g., Gentzen (1935), Dummett (1991), Prawitz (2006), Martin-Löf (1996), and, as far as logical and mathematical vocabulary is concerned, Tennant (1997)). Other options are, however, available: e.g., elimination rules can be privileged (see, e.g., Oliveira (2021), this view is also discussed in Dummett (1991) despite privileging the introduction view), or neither introduction nor elimination rule can be privileged (see, e.g., De Queiroz (2008)).

[4] See Tennant (1999).

[5] See, e.g., Steinberger (2009).

[6] See, e.g., Rinaldi et al. (2018), Derakhshan and Pfenning (2020), UFP (2013).

[7] For simplicity, we will consider only $A \vee \perp$ form of a disjunction, i.e., A as the left disjunct and $\perp$ as the right disjunct, throughout this paper. However, no difference arises if we switch them, i.e., we take $\vee$ to be commutative.

$$\frac{\perp}{A}$$

which is the well-known EFQ rule.

As we will show, the collapse rule is deductively just as strong as EFQ but it has one major advantage over EFQ when considered as a basic rule: the explosion principle is not a stipulated property of $\perp$ but a proof-theoretic consequence of the inferential interaction between $\vee$ and $\perp$. Furthermore, the collapse rule can also be used to derive the disjunctive syllogism (DS), which can be viewed as an extended version of the collapse rule relying on the notion of contradiction instead of absurdity. To emphasize, the purpose of this paper is not to reject EFQ or the explosion principle it directly captures, but to consider an alternative justification where it is not regarded as a basic rule of the system but as a derived rule, i.e., a rule justified by the other rules of the system.

As a background framework, we assume natural deduction for intuitionistic propositional logic (IPC; Gentzen (1935), Prawitz (1965)) as it is sufficient to demonstrate all the main ideas.[8] Furthermore, these ideas can be carried over to classical logic (IPC + classical reductio ad absurdum rule) without change. The language $\mathcal{L}$ of IPC is a set of formulas $A, B, C, \ldots$ built in the usual way from atomic formulas (propositional variables) $p, q, r, \ldots$, binary connectives $\wedge$, $\vee$, and $\rightarrow$ for conjunction, disjunction, and implication, and propositional constant $\perp$ for absurdity (with negation defined as $A \rightarrow \perp$), which will be understood as a nullary disjunction $\vee_0$ expecting no propositions. Additionally, we will also introduce an unary disjunction $\vee_1$ that expects a single proposition.[9]

This paper is structured as follows. In Section 2, we introduce the idea of treating absurdity as a nullary disjunction, in Section 3, we present the informal interpretation of disjunction as an incomplete communication and apply it to the case of nullary disjunction, in Section 4 we provide informal and formal justifications for the collapse rule, and finally in Section 5 we explore the relation of the collapse rule to the disjunctive syllogism (DS) and to the ex falso quodlibet rule (EFQ).

[8]One of the reviewers inquired whether it would be possible to adopt the collapse rule into sequent calculus as well. Yes, it can be done. For example, assuming multiple-conclusion sequent calculus, the collapse rule could be understood as a natural counterpart to the structural rule of weakening on the right. Specifically, it would become a structural rule of "strengthening" on the right allowing us to derive a sequent $\Gamma \Rightarrow A$ from $\Gamma \Rightarrow A, \perp$. However, as the topic of sequent calculus is beyond the scope of the present paper (including considerations about the cut-elimination theorem), we will not pursue it further here.

[9]As mentioned by one of the reviewers, we could regard $\vee_0$ and $\vee_1$ not as separate connectives but just as distinct cases of disjunction based on the number of different arguments/propositions they received. That is generally true, however, we want to be more explicit about it and presuppose that the arity of each logical connective is set in advance to a particular number at the time of definition of the connective in question. This also allows us to more easily apply the "disjunction-as-incomplete communication" informal interpretation discussed in Section 3.

2 $\bot$ as a nullary disjunction

Before we move on to the justification of the collapse rule, let us briefly consider the notion of $\bot$ we will be relying on. As we have mentioned in the beginning, we will adopt the logical approach towards $\bot$. Specifically, we will regard $\bot$ as a nullary, i.e., a zero-place disjunction connective.[10] On this approach, $\bot$ effectively becomes a null choice, i.e., a degenerate variant of a choice with no alternatives. This is a common practice, especially in more computer science and type-theoretic-oriented literature.[11] To better emphasize this viewpoint, let us temporarily use the symbol "$\vee_0$" instead of "$\bot$". Thus, the collapse rule will look as follows:

$$\frac{A \vee \vee_0}{A}\,\text{collapse}$$

It effectively expresses the idea that a choice between A and having no choice is not really a choice at all and it collapses to the only option, i.e., A.

What is the reasoning behind identifying $\bot$, commonly understood as absurdity, contradiction, or falsity, with the nullary disjunction $\vee_0$? To properly explain this, let us start by considering (binary) disjunction. Since $A \vee B$ is true if A or B is true, there are two ways to prove it, which are expressed by the following two introduction rules:

$$\frac{A}{A \vee B}\,\vee\mathrm{I}_l \qquad\qquad \frac{B}{A \vee B}\,\vee\mathrm{I}_r$$

And since there are two introduction rules, the corresponding elimination rule has two cases to consider:

$$\frac{A \vee B \qquad \begin{matrix}[A]\\ C\end{matrix} \qquad \begin{matrix}[B]\\ C\end{matrix}}{C}\,\vee\mathrm{E}$$

and it captures the mechanism of proof by cases. Intuitively, even if we know that the major premise $A \vee B$ of this elimination rule is true, we still do not know whether A or B is true, so if want to eliminate the disjunction, we have to check both cases. If C follows from both of them, then we can safely eliminate $A \vee B$ and derive C.

Now, let us consider the absurdity constant $\bot$. We assume $\bot$ to have no introduction rule since there should be no way to prove absurdity (and thus make it true), and a single elimination rule also known as the ex falso quodlibet rule (EFQ) (see, e.g., Gentzen (1935), Prawitz (1965), Martin-Löf (1984), Troelstra and Schwichtenberg (2000)):

$$\frac{\bot}{C}\,\mathrm{EFQ}$$

[10] See, e.g., Humberstone (2011), p. 440.

[11] See, e.g., Granström (2011), p. 38; Rinaldi et al. (2018), p. 227; UFP (2013), p. 33, Schultz and Spivak (2019), p. 103, Derakhshan and Pfenning (2020), p. 4.

Observe that the same general reasoning as with disjunction can be applied here: with disjunction, we had two introduction rules corresponding to two cases in the elimination rule; with $\perp$, we have no introduction rule, so we should have no corresponding cases to consider – and indeed we don't. The justification of this rule is a little bit more tricky: by using this rule we are effectively saying that we can prove that a proposition C is true once we are provided with a proof that $\perp$ is true. But making such a promise is easy as we know that it will never come up: since there is no proof of $\perp$ by definition, it is impossible to be ever provided with a proof that $\perp$ is true, and thus we will never actually have to show how to prove that the proposition C is true.[12] As Martin-Löf (1996) (p. 52) explains: "The undertaking that you make when you infer by [EFQ] is therefore like saying,

I shall eat up my hat if you do such and such,

where such and such is something of which you know, that is, are certain, that it cannot be done."

Note that from this perspective, EFQ can be regarded as justified by the semantic principle of vacuous truth. Specifically, we can say that EFQ corresponds to the conditional "if $\perp$ is true, then C is true" which is vacuously true because $\perp$ can never be true (by definition). In other words, the justification of the EFQ rule seems to rest on the vacuous truth principle which in turn rests on the definition of the conditional.[13]

This observed analogy between introduction and elimination rules for $\perp$ and disjunction is then taken as a justification for treating $\perp$ as a nullary disjunction. Thus, on this approach, EFQ becomes essentially just a degenerate case of the disjunction elimination rule with zero disjuncts/cases to consider:[14]

$$\dfrac{\cancel{A} \vee_0 \cancel{B} \qquad \begin{matrix}\cancel{[A]} \\ \cancel{C}\end{matrix} \qquad \begin{matrix}\cancel{[B]} \\ \cancel{C}\end{matrix}}{C}$$

which is, of course, nothing other than the EFQ rule:

$$\dfrac{\vee_0}{C}$$

[12]See, e.g., Martin-Löf (1996), Sommaruga (2000).

[13]This, of course, opens the issue of the definition of the conditional. And although the requirement of justification is usually not associated with definitions, which are after all just stipulations (just as introduction rules), we should still be able to motivate or explain them. In that case, it would be interesting to explore why we define conditionals as vacuously true when $\perp$ is the antecedent. However, such an investigation is beyond the scope of the present paper. It is also worth emphasizing that the vacuous truth principle is not exclusive to classical logic and applies, e.g., to intuitionistic logic as well. We thank one of the reviewers for bringing this matter to our attention.

[14]Of course, this approach does not really solve all the issues surrounding absurdity, e.g., it still makes sense to ask the question why anything should follow from a nullary disjunction (i.e., why should EFQ hold). And, arguably, answering this question in a philosophically satisfactory way seems even harder than for the case when $\perp$ is considered absurdity. The answer along the lines of "EFQ should hold because we know that $\vee$E holds and EFQ is what remains from $\vee$E when we have zero cases to consider" does not seem particularly illuminating.

3 $\vee$ as an incomplete communication

In intuitionistic and constructive tradition, it is not uncommon to regard disjunction as an incomplete communication (see, e.g., Hilbert and Bernays (1968), Kleene (1945)). This informal interpretation was especially popularized by Kleene in connection to his realizability semantics:

> [a disjunction $A \vee B$ is] an incomplete communication, to complete which one must, either indicate that "A" is to be completed and supply such information as may be needed to complete "A", or indicate that "B" is to be completed and supply such information as may be needed to complete "B". (Kleene (1973), p. 99)

In other words, disjunction $A \vee B$ is supposed to represent a choice between options A or B. And when we remove the options, we remove its constructive character as well. As van Dalen describes it:

> Disjunction intuitively calls for a decision: which of the two disjuncts is given or may be assumed? This constructive streak of $\vee$ is crudely but conveniently blotted out by the identification of $\varphi \vee \psi$ and $\neg(\neg\varphi \wedge \neg\psi)$. The latter only tells us that φ and ψ cannot both be wrong, but not which one is right. (van Dalen (2013), pp. 50-51)

In the previous Section 2 we have shown that we can regard $\perp$ as a nullary disjunction $\vee_0$. But that means that from the communication perspective presented above, there is nothing to indicate with $\vee_0$, as it has no disjuncts to choose from, nor there is anything to supply, as $\vee_0$ has no proofs. In other words, with nullary disjunction $\vee_0$, there is no decision to be made and we already know that it cannot be right. Simply put, in a consistent system, there is no way to complete the communication $\vee_0$, it is an *incompletable* communication. Observe, however, that this realization alone makes it, somewhat paradoxically, a complete communication. In other words, if we assert $\vee_0$, no further information is needed since we already know what $\vee_0$ communicates by its very definition (i.e., something that cannot be proved), and this reflection completes it as a communication. Simply put, there is nothing incomplete about communicating the incompletable communication $\vee_0$.

But doesn't that also mean that $\vee_0$ cannot be understood as disjunction after all as it behaves differently from $\vee$ under the communication interpretation? Not necessarily, it just means that $\vee_0$ is a degenerate case of disjunction and as such it exhibits properties different from the nondegenerate one. Thus, just as, e.g., a zero-dimensional point can be considered a degenerate case of a two-dimensional circle when the size of the radius is 0, so can a zero-option disjunction (and thus a complete communication) be considered a degenerate case of a two-option disjunction (an incomplete communication) when the number of disjuncts is 0.

Remark 1. The constructive nature of a propositional system is often tied to the behavior of disjunction (see, e.g., Troelstra (1973), p. 91, van Dalen (2013), p. 161). A test for constructivity then consists of checking whether or not the system at hand enjoys the disjunction property: if $A \vee B$ is provable then A is provable or B is provable.[15] As expected, it holds in intuitionistic logic but fails in classical logic. Furthermore, it is known that if a formula A of an intuitionistic propositional logic does not contain $\vee$ (or it is restricted to some specific positions, as with, e.g., Harrop formulas[16]), it behaves classically. More specifically, the principle of double negation $A \equiv \neg\neg A$ applies to it, where the symbol "$\equiv$" denotes logical equivalence, i.e., that $\neg\neg A$ can be derived from A and vice versa. And it is clear that $\vee_0$ does not contain $\vee$, as it contains only itself, i.e., $\vee_0$ (recall that $\vee_0$ is an alternative symbol for $\perp$).

Note that what we have said about $\vee_0$ also extends towards the disjunction of the form $A \vee \vee_0$. Due to the appearance of $\vee_0$, it also cannot be any longer considered to be incomplete communication or presenting a choice in the same sense as $A \vee B$. Analogously to $\vee_0$, when we are presented with $A \vee \vee_0$ there is nothing really to indicate when we are concerned with its provability, as only one disjunct can be made true and the other is constantly false. In other words, there is no choice in deciding which of the disjuncts of $A \vee \vee_0$ is right, as one of them is always wrong. However, in contrast to $\vee_0$, there is something to supply, specifically the proof for the disjunction $A \vee \vee_0$ as disjunctions have introduction rules, i.e., they can be proved. The proof we must supply has to be, of course, the proof for the left disjunct A as there are no proofs for $\vee_0$ by definition (recall that absurdity has no introduction rules). So, in this sense, $\vee_0$ is less incomplete communication than $A \vee \vee_0$ which is then less incomplete than $A \vee B$. Thus, $A \vee \vee_0$ is rather a "semi-incomplete" communication (this observation will become useful in Section 4.2.2).

We have seen that if we remove both options A and B from the choice $A \vee B$ (i.e., introduce the nullary disjunction $\vee_0$), it ceases to be an incomplete communication in Kleene's sense. But what happens if we remove just one of the options? In other words, what happens if we introduce a unary disjunction $A \vee_1$?

First, let us consider what would be the introduction and elimination rules for such a disjunction. Since there is only one disjunct, there is only one way to prove it. Thus, the introduction rule will be as follows:

$$\frac{A}{A \vee_1} \vee_1\mathrm{I}$$

And because there is only a single introduction rule for $A \vee_1$, there will be only a single case to consider in the corresponding elimination rule:

[15] Viewing disjunction property as the defining feature of every constructive system is not without its problems, see, e.g., Troelstra and van Dalen (1988), p. 139. For an opposing view on the importance of constructive disjunction, see, e.g., Hazen (1990). We thank one of the reviewers for this reference.

[16] Harrop formulas (see Harrop (1956), also known as Rasiowa-Harrop formulas, Rasiowa (1954)), are formulas that do not contain $\vee$ except in the antecedents of implications. For interesting properties of Harrop formulas from the perspective of constructive validity of rules, see Pezlar (2024).

$$\dfrac{A \vee_1 \qquad \begin{matrix}[A] \\ C\end{matrix}}{C} \vee_1\mathrm{E}$$

Note that if C is A again in $\vee_1$E, we can consider a simplified variant:

$$\dfrac{A \vee_1}{A} \vee_1\mathrm{E}'$$

as from A we can obtain, of course, A again.

Now, it should be clear that once we remove a single option from the choice $A \vee B$, it is no longer an incomplete communication. Analogously to $A \vee \vee_0$, we can say that having a choice with only one option is not really a choice at all.[17] Thus, from this perspective, it seems unproblematic to identify $A \vee_1$ with just A. In other words, the unary disjunction $A \vee_1$ and A are logically equivalent, i.e., $A \vee_1 \equiv A$ (we will establish this later in Section 4.2.1).[18]

So, to sum up, the "disjunction-as-incomplete communication" informal interpretation allows us to make sense of the difference between nullary $\bot$ ($= \vee_0$), unary $\vee_1$, and binary $\vee$ disjunctions in terms of the amount of information they communicate. In other words, not all disjunctions are incomplete communications in the same sense: some communicate more, some less. But most importantly, this interpretation allows us to show that there is actually a difference in the information communicated between $A \vee \bot$ and $A \vee B$, even though they are both binary disjunctions.

The key observation for this is that a nullary disjunction $\bot$ is no longer an incomplete communication, which in turn helps us explain why $A \vee \bot$ communicates more information than $A \vee B$. The reason is that from the communication perspective, $A \vee \bot$ effectively communicates the same information as a unary disjunction $A \vee_1$, i.e., a disjunction with one disjunct removed. Specifically, in both cases, there is nothing to indicate (recall Kleene's quote from earlier) as only A can be true and in both cases, there is something to supply, i.e., the corresponding proofs for $A \vee \bot$ and $A \vee_1$. Once we adopt the collapse rule, we can actually show that $A \vee \bot$ and are $A \vee_1$ logically equivalent, i.e., $A \vee \bot \equiv A \vee_1$, see Section 4.2.1. In the next section, we will utilize the difference in information content between $A \vee B$ and $A \vee \bot$ to offer an additional informal justification for the collapse rule.

4 Justification of the collapse rule

Although the main purpose of this paper is not to justify the collapse rule itself but rather to show that it can be used to justify EFQ and the disjunctive

[17] Note, however, that there is a crucial difference between $A \vee_1$ and $A \vee \vee_0$. While the former removes an option entirely, the latter replaces it with an explicit null option, i.e., an option that cannot be chosen. From this perspective, $\vee_0$ can be seen as an object-level representation of a missing option.

[18] It is clear that unary disjunction $A \vee_1$ is very naturally equated with just asserting the proposition A. If we are given a choice with just a single option, there is, of course, only one possible outcome of the choice, and thus, there is no choice at all. Note that this is analogous to the reasoning we used when explaining the collapse rule.

syllogism (DS), it is prudent to show how we can justify it as well since the whole argument relies on it.

4.1 The status of the collapse rule

The collapse rule:

$$\frac{A \vee \bot}{A}\ \text{collapse}$$

can be derived with the help of EFQ (and disjunction elimination) as follows:

$$\dfrac{A \vee \bot \qquad [A]^1 \qquad \dfrac{[\bot]^2}{A}\,\text{EFQ}}{A}\ \vee\mathrm{E}_{1,2}$$

but this way of justification is not feasible because, as we have mentioned, we want to use the collapse rule to justify EFQ, not vice versa. Thus, we will have to approach the justification of the collapse rule differently.

What options are there? Generally speaking, they are all linked to the question of what kind of rule is the collapse rule. From what we have said above, it is clear that it cannot be a structural role as it deals with logical connectives. So, from this perspective, it is safe to say that it is a logical rule. But what kind of a logical rule?

The characteristic logical rules of natural deduction are introduction and elimination rules (Gentzen (1935), Prawitz (1965)). First, to simplify the discussion a little, let us assume that the collapse rule is a single-ended rule in Dummett's sense, i.e., that it is "either an introduction rule but not an elimination rule, or an elimination rule but not an introduction rule" (Dummett (1991), p. 256).[19] Upon inspecting the rule, it is clear again that it cannot be an introduction rule for a specific connective since no new connective is introduced in the conclusion.[20] So, the collapse rule seems to be an elimination rule. But what does it eliminate? There are two logical connectives in the premise that disappear in the conclusion: disjunction $\vee$ (which is the main connective) and $\bot$ (the right disjunct).[21] So, it seems we can view it as either an elimination

[19] Note, however, that we can regard the collapse rule as completely outside of the introduction/elimination rules framework, similarly as is disjunctive syllogism (DS), and our points from Sections 4.2.1 and 4.2.2 would still apply with the exception of the justification based on harmony in Section 4.2.3, which presupposes that the collapse rule is regarded at least as an elimination rule.

[20] One of the reviewers suggested that we could perhaps view the collapse rule as an unspecific introduction rule for A's main connective, assuming A is not an atomic proposition. That is true but it would also mean that the main connective to be introduced (and thus defined) was already present (and thus presupposed) in the premise which is generally problematic from the verificationist viewpoint of introduction rules as the meaning-giving rules (see, e.g., Pezlar (2021)).

[21] Typically, the elimination rules operate only on the main connectives (see, e.g., Prawitz (1965), Negri and von Plato (2001), Mancosu et al. (2021)), but here we utilize more general notion.

rule for $\vee$, or an elimination rule for $\bot$ (recall that $\bot$ is an alternative notation for $\vee_0$). Let us examine both these options. We begin with the first one.

Collapse rule as a (binary) disjunction elimination. Since disjunction is the main connective of the premise of the collapse rule and it gets eliminated in the conclusion, it seems most natural to regard it as a disjunction elimination rule. The slight complication with this view is that we already have an elimination rule for disjunction: the $\vee$E rule. Thus, if we want to keep this view, we have to explain in what relation is the collapse rule to $\vee$E. A most straightforward way seems to be to regard the collapse rule simply as a variant of the disjunction elimination rule when the disjunction to be eliminated has the form $A \vee \bot$. There is, however, a problem with this approach. Specifically, the collapse rule can be regarded as a particular case of the disjunction elimination rule $\vee$E only in the presence of EFQ (recall the derivation at the beginning of this section). Once EFQ is absent, or taken as a rule in need of a justification, the collapse rule cannot be reduced to the disjunction elimination rule $\vee$E as there is no way to complete its second subcase, i.e., to get from $\bot$ to A without EFQ, and thus ultimately to get from $A \vee \bot$ to A via $\vee$E. Once the collapse rule is adopted, the derivation goes through:

$$\dfrac{A \vee \bot \qquad [A]^1 \qquad \dfrac{\dfrac{[\bot]^2}{A \vee \bot}\,{\scriptstyle \vee\mathrm{I}_r}}{A}\,{\scriptstyle \text{collapse}}}{A}\,{\scriptstyle \vee\mathrm{E}_{1,2}}$$

In other words, without EFQ the collapse rule and the disjunction elimination rule $\vee$E have to be considered as distinct basic rules for disjunction elimination.

Collapse rule as an absurdity (nullary disjunction) elimination. This view is possible only if we allow elimination rules to affect even non-main connectives. From this perspective, the rule can be understood as codifying the principle "If there is a way to avoid absurdity (nullary disjunction), always take it". In other words, if there is a choice between having an option and having no option at all, always prefer having an option. Thus, we can regard it as a way to escape from an optionless situation, similar to EFQ which allows us to escape from an absurd situation.

In this paper, we will prefer the first option, i.e., treating the collapse rule as a specific variant of a disjunction elimination rule. This seems to be the most natural choice given that disjunction is the main connective of its premise. Furthermore, it will allow us to analyze the collapse rule using the notion of harmony w.r.t. disjunction introduction rules (see Section 4.2.3).

Now, let us finally move on to the justification of the collapse rule.

4.2 Justifying the collapse rule

4.2.1 The collapse rule as a derived rule

Let us repeat for convenience the collapse rule:

$$\frac{A \vee \bot}{A}\ \text{collapse}$$

Now, taking into consideration all that we have said above (specifically, $\bot \equiv \vee_0$, $A \equiv A\vee_1$, and writing $\vee$ as $\vee_2$ to emphasize that it is the standard binary disjunction), we can also write it as:

$$\frac{A \vee_1 \vee_2 \vee_0}{A}\ \text{collapse}$$

but for reasons of clarity, we will keep the more common notation from above and suppress the arity indexes when possible.

To justify this rule all we need to do is to show that the disjunction $A \vee \bot$ implies A. We can, however, go even further: we can show that the premise is logically equivalent to the conclusion, i.e., that $A\vee\bot \equiv A$, which is a well-known equivalence. In other words, we can show that a binary choice between a unary choice and a nullary choice is the same as that unary choice (i.e., A).

To simplify the task, we can split it into two subgoals:

1. showing that $A \vee \equiv A$, and
2. showing that $A \vee \bot \equiv A \vee$.

Subgoal 1. To establish $A \vee \equiv A$, we want to show that A implies $A\vee$ and vice versa. This is straightforward and we can demonstrate it with the rules for $\vee_1$ and implication introduction. Specifically, the first direction can be established as follows:

$$\dfrac{\dfrac{[A]^1}{A \vee}\ \vee_1\mathrm{I}}{A \to (A \vee)}\ {\to}\mathrm{I}_1$$

And the other direction:

$$\dfrac{\dfrac{[A \vee]^1 \qquad [A]^2}{A}\ \vee_1\mathrm{E}_2}{(A \vee) \to A}\ {\to}\mathrm{I}_1$$

Intuitively, this also makes sense: a single-option choice A is the same as A (i.e., no choice takes place).

Subgoal 2. First, with the result established in subgoal 1, we can simplify

$$A \vee \bot \equiv A \vee$$

to

$$A \vee \bot \equiv A$$

Now, to establish this, we want to show that A implies $A \vee \bot$ and vice versa. First, let us establish the direction from A to $A \vee \bot$. This is straightforward and can be demonstrated as follows:

$$\dfrac{\dfrac{[A]^1}{A \vee \bot}\,\vee\mathrm{I}_l}{A \to (A \vee \bot)}\,\to\mathrm{I}_1$$

Now, the other direction is more problematic. If we were allowed to use EFQ, this would be easy:

$$\dfrac{\dfrac{[A \vee \bot]^1 \qquad [A]^2 \qquad \dfrac{[\bot]^3}{A}\,\mathrm{EFQ}}{A}\,\vee\mathrm{E}_{2,3}}{(A \vee \bot) \to A}\,\to\mathrm{I}_1$$

However, EFQ is off the table (analogously for DS), thus we will have to find a different route. Specifically, we want to find a way to get from $A \vee \bot$ to A. In other words, we want to construct a derivation of the following shape:

$$\dfrac{\begin{matrix}[A \vee \bot]^1 \\ ? \\ A\end{matrix}}{(A \vee \bot) \to A}\,\to\mathrm{I}_1$$

and we are looking for a derivation that would replace the placeholder "?". We already know that A can be derived from $\vee A$, so let us try to fill it in:

$$\dfrac{\dfrac{\begin{matrix}[A \vee \bot]^1 \\ ? \\ A \vee\end{matrix} \qquad [A]^2}{A}\,\vee_1\mathrm{E}_2}{(A \vee \bot) \to A}\,\to\mathrm{I}_1$$

But now, we are stuck again: we have no clear way to derive $\vee A$ from $A \vee \bot$.

Of course, the most straightforward route to get from $A \vee \bot$ to $\vee A$ is to introduce an additional rule:

$$\dfrac{A \vee \bot}{A \vee}\,\mathrm{R}$$

which then allows us to complete the derivation:

$$\dfrac{\dfrac{\dfrac{[A \vee \bot]^1}{A \vee}\,\mathrm{R} \qquad [A]^2}{A}\,\vee_1\mathrm{E}_2}{(A \vee \bot) \to A}\,\to\mathrm{I}_1$$

Note, however, that the rule R is nothing other than the collapse rule again with A replaced by $A\vee$ (recall that $A \equiv A\vee$), i.e.,

$$\dfrac{A \vee \bot}{A \vee}\,\mathrm{R} \quad = \quad \dfrac{A \vee \bot}{A}\,\text{collapse}$$

Thus, trying to establish the collapse rule (without invoking EFQ or DS) does not seem possible without using the collapse rule itself. This brings the derivability justification process to a natural conclusion: we either have to accept the collapse rule or not. Of course, this result was to be expected since we have taken the collapse rule as a basic rule of the system.

4.2.2 Informal justification

Discovering that the collapse rule cannot be derived using other rules of the system, unless we also accept EFQ or DS, does not, however, mean that the rule cannot be justified informally. This is, after all, the case with all the basic rules such as conjunction introduction, implication elimination, etc. We cannot analyze them any further, we can only explain them (see, e.g., Martin-Löf (1984)). To borrow a term from Martin-Löf (1996), we could say that the collapse rule is a rule of *immediate inference*: we have to make the conclusion of the rule immediately evident to us, assuming we already know its premise is true. With the collapse rule, this does not seem problematic. If we know that $A \vee \bot$ is true and that $A \vee \bot$ is true if one of its disjuncts is true and that $\bot$ cannot be true, then it seems immediately evident that it is the case that A has to be true. In contrast to EFQ's explosion principle, let us call this the implosion principle, i.e., $A \vee \bot$ "imploding" into A.[22]

Someone might argue that this justification is no better than the one typically associated with EFQ which goes as follows: since we know that $\bot$ cannot be true, if it is true, then any A is true as well. As we mentioned earlier, this justification can be seen as based on the vacuous truth principle, and thus ultimately on our understanding of the conditional at play. In our opinion, the justification of EFQ based on the collapse rule is better in the following way: We can reject EFQ by simply not accepting the conditional "if $\bot$ is true, then any A is true as well" (i.e., by directly rejecting the explosion principle, and thus also the understanding of the conditional that produces it). After all, it seems quite reasonable to stop at the point of deriving absurdity and not draw any more conclusions (and, again, historically this was many times the case, see, e.g., Hewitt (2022)). On the other hand, rejecting the conditional "if $\bot$ is true, then any A is true as well" does not lead outright to the rejection of the collapse rule (or the explosion principle it produces). More specifically, the collapse rule, in contrast to EFQ, does not rely on the semantic principle of vacuous truth, i.e., on a specific understanding of the conditional that allows it: the explosion is a result of a purely proof-theoretic interaction between the collapse rule and disjunction introduction rule (this will be explored more formally in Section 4.2.3). Thus, from this perspective, the justification of the collapse rule is conceptually simpler.

As mentioned above, the collapse rule can be also justified from the perspective of the constructive "disjunction-as-incomplete communication" informal interpretation. The reasoning is as follows. In general, derivations such

[22]Note the similarity of this reasoning to the reasoning associated with the disjunctive syllogism, which will be discussed later.

as:

$$\frac{A \vee B}{A}$$

are invalid because $A \vee B$ is an incomplete communication: just from $A \vee B$ alone we cannot decide whether A or B should be derivable from it. In contrast, derivations such as:

$$\frac{A \vee \bot}{A}$$

are no longer invalid because $A \vee \bot$ is no longer an incomplete communication in the same sense as $A \vee B$, as shown in Section 3. As discussed there, the binary disjunction $A \vee \bot$ between A and nullary disjunction $\bot$ (= absurdity) effectively communicates the same amount of information as disjunction with a single disjunct removed, i.e., the unary disjunction $A \vee_1$. This is because with both $A \vee \bot$ and $A \vee_1$ there is no need to indicate which of the disjunct has to be completed, as only one can be, i.e., A, and simultaneously with both $A \vee \bot$ and $A \vee_1$ there is a need to supply a proof for them. And since $A \vee_1$ can be identified with A as also discussed in Section 3 (formally, this is established in Section 4.2.1), we can conclude that from $A \vee \bot$ we can derive A. From this perspective, we can regard the collapse rule as a rule for making certain communications, specifically those semi-incomplete communications of the form $A \vee \bot$ "more complete" by reducing them just to A.

Even intuitively, the collapse rule does make sense: it is essentially a choice simplification rule that says that a choice between A and having no choice is no choice at all and can be simplified to A.

Remark 2. It is worth noting that the implosion principle is not just a formal contrivance and support for it can also be found in natural language. More specifically, the collapse rule can be regarded as capturing the principle behind English (and other languages) idioms such as:

> "*It's my way or the highway* [which I know you don't really want to take]."[23]

i.e., a choice between two options – one of which is not really an acceptable one – defaults to the only acceptable one. In other words, it captures the idea that sometimes we are given a choice that is not really a choice as it has only one possible outcome. For example:

- Our high school football coach tells us that we can either do it the way they want or we can leave the team.

[23] It is true that similar points can also be made about EFQ. For example, we can regard it as capturing the principle behind phrases such as "[if something I am sure is false is true] *I'll eat my hat*." These natural language observations are, however, not made to address the question of the validity of these principles but rather the question of their artificiality. From this perspective, the collapse rule seems to be just as natural as the EFQ rule.

- Our boss tells us that if we don't like the way they run the company we can quit.
- A hijacker of a cruise ship tells us in the middle of the ocean that we can either stay or jump off the ship.

Technically speaking, all of these are perfectly good choices, but in practice not so much. And similar applies to the choice $A \vee \perp$: of course, we can choose $\perp$ instead of A, i.e., end up in absurdity when we had an option not to, but in practice this is not really an option anyone would be willing to take.

4.2.3 Harmony and the explosion principle

We have provided an informal justification for the collapse rule based on the idea that it is a rule of immediate inference. However, since we are treating the collapse rule as an elimination rule (specifically for disjunction), there is still one more route of possible formal justification for the collapse rule that is left unexplored.

Earlier, we have said that elimination rules should be justifiable w.r.t. their corresponding introduction rules. Another way of putting this is that the elimination rules should be in harmony (see Dummett (1991)) with the introduction rules which means that the elimination rules should not allow us to derive *more* or *less* than the introduction rules permit.[24]

To determine whether this kind of balance obtains between the introduction and elimination rule, we can check whether the elimination rules in question satisfy two properties (see Pfenning and Davies (2001), also Jacinto and Read (2017)): *local soundness* (= showing that the elimination rules are not too strong, i.e., that they do not allow us to derive more than the introduction rules permit) and *local completeness* (= showing that the elimination rules are not too weak, i.e., that they do not allow us to derive less than the introduction rules permit).

To check that local soundness holds, we have to show that whatever conclusion can be obtained by applying the elimination rule in question to the result of the corresponding introduction rule, can be already obtained by a more direct means without the need to invoke the elimination rule. In other words, we have to demonstrate that the application of the introduction rule followed immediately by the application of the corresponding elimination rule constitutes, in a way, an unnecessary detour in the derivation (it brings no new information) and can be removed.

[24] Some authors (e.g., Francez (2015)) use the term "harmony" to refer only to the property that the elimination rules should not allow us to derive more, and use the term "stability" to refer to the property that the elimination rules should not allow us to derive less (i.e., harmony and stability are understood as dual properties). To complicate the terminological issue further, some authors (see, e.g., Kürbis (2019), Jacinto and Read (2017)) use "stability" in the sense we use here the term "harmony", i.e., encompassing both the no more and no less aspects. For our usage of the term, see, e.g., Pfenning and Davies (2001), Tranchini (2021), Schroeder-Heister (2014).

Formally, this is achieved by presenting a local reduction rule that shows us how to remove this kind of detour. For example, in the case of disjunction, we obtain the following two reduction rules (since we have two introduction rules):

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ A\end{matrix}}{A \vee B}\vee\mathrm{I}_l \qquad \begin{matrix}[A]\\ \mathcal{D}_2\\ C\end{matrix} \qquad \begin{matrix}[B]\\ \mathcal{D}_3\\ C\end{matrix}}{C}\vee\mathrm{E} \quad \xRightarrow[\vee\text{-red}_1]{\text{reduces to}} \quad \begin{matrix}\mathcal{D}_1\\ A\\ \mathcal{D}_2\\ C\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ B\end{matrix}}{A \vee B}\vee\mathrm{I}_r \qquad \begin{matrix}[A]\\ \mathcal{D}_2\\ C\end{matrix} \qquad \begin{matrix}[B]\\ \mathcal{D}_3\\ C\end{matrix}}{C}\vee\mathrm{E} \quad \xRightarrow[\vee\text{-red}_2]{\text{reduces to}} \quad \begin{matrix}\mathcal{D}_1\\ B\\ \mathcal{D}_3\\ C\end{matrix}$$

where $\mathcal{D}_1, \mathcal{D}_2, \ldots \mathcal{D}_n$ represent derivations (with possibly open assumptions).

To establish that local completeness holds, we have to show that the application of the elimination rules in question to some proposition leads to a conclusion from which we can always derive back the original proposition via the corresponding introduction rules. In other words, we have to demonstrate that the application of the elimination rule loses no information and from its conclusion, we can rederive the just eliminated proposition. Thus, we are, in a way, unnecessarily expanding the original derivation.

Formally, this is achieved by presenting a local expansion rule that shows us how to construct this kind of expansion. For example, in the case of disjunction, we obtain:

$$\begin{matrix}\mathcal{D}\\ A \vee B\end{matrix} \quad \xRightarrow[\vee\text{-exp}]{\text{expands to}} \quad \dfrac{\begin{matrix}\mathcal{D}\\ A \vee B\end{matrix} \qquad \dfrac{[A]}{A \vee B}\vee\mathrm{I}_l \qquad \dfrac{[B]}{A \vee B}\vee\mathrm{I}_r}{A \vee B}\vee\mathrm{E}$$

Now, since we were able to exhibit the required local reduction (i.e., $\vee$-red$_1$ and $\vee$-red$_2$) and local expansion (i.e., $\vee$-exp) for $\vee$E, and thus establish its local soundness and local completeness, respectively, we can claim that the elimination rule $\vee$E for disjunction is harmonious, i.e., it is in a balance with the corresponding introduction rules.

Can we do the same with the collapse rule? More specifically, if we treat the collapse rule as a disjunction elimination rule, can we show it to be harmonious with the disjunction introduction rules?

Let us start by considering the property of local completeness, i.e., whether the collapse rule is not too weak as an elimination rule for disjunction $A \vee \bot$. In other words, we have to check if $A \vee \bot$ can be rederived using consecutive applications of the collapse rule (understood as a disjunction elimination rule) and disjunction introduction rule. To demonstrate this, we have to provide a rule for local expansion. This can take the following form:

$$\begin{matrix}\mathcal{D}\\ A \vee \bot\end{matrix} \quad \xRightarrow[\text{collapse-exp}]{\text{expands to}} \quad \dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ A \vee \bot\end{matrix}}{A}\text{collapse}}{A \vee \bot}\vee\mathrm{I}_l$$

Thus, we can claim that the collapse rule is locally complete.

Now, let us consider the local soundness, i.e., whether the collapse rule is not too strong as an elimination rule for disjunction $A \vee \bot$. In other words, we have to check if $A \vee \bot$ does not allow us to derive more than the corresponding introduction rules permit by creating detours that cannot be removed. To demonstrate this, we have to provide rules for local reduction. Let us start by assuming that $A \vee \bot$ was derived from A, i.e., we obtain the following derivation:

$$\mathcal{D}_A\colon \qquad \cfrac{\cfrac{\begin{matrix}\mathcal{D}\\ A\end{matrix}}{A \vee \bot}\,\vee\mathrm{I}_l}{A}\,\text{collapse}$$

As we can see, in this case, the application of disjunction introduction $\vee\mathrm{I}_l$ followed by the collapse rule really constitutes an unnecessary detour: if we remove it, we end up where we started, i.e., with the derivation of A. Nothing is gained from the detour $\mathcal{D}_A$. The corresponding reduction would then look as follows:

$$\cfrac{\cfrac{\begin{matrix}\mathcal{D}\\ A\end{matrix}}{A \vee \bot}\,\vee\mathrm{I}_l}{A}\,\text{collapse} \qquad \xRightarrow[\text{collapse red}_1]{\text{reduces to}} \qquad \begin{matrix}\mathcal{D}\\ A\end{matrix}$$

Now, let us assume that $A \vee \bot$ was derived from $\bot$, i.e., we obtain the following derivation:

$$\mathcal{D}_\bot\colon \qquad \cfrac{\cfrac{\begin{matrix}\mathcal{D}\\ \bot\end{matrix}}{A \vee \bot}\,\vee\mathrm{I}_r}{A}\,\text{collapse}$$

First, note that the derivation $\mathcal{D}_\bot$ is the same derivation as we have used at the beginning of this paper to show that EFQ can be considered as a derived rule once we adopt the collapse rule. In other words, this derivation demonstrates how the collapse rule indirectly captures the explosion principle, i.e., it shows us how to get from a derivation of $\bot$ to an arbitrary proposition A.

Inadvertently, however, what this also means is that what we have constructed here is not an unnecessary detour because if we remove it, we do not end up where we started. Specifically, we have started with the derivation of $\bot$ but ended with a derivation of A. To put it differently, to be able to remove this "detour", we would need to be able to get from $\bot$ to A, i.e., we would need an explosion, without the use of the collapse rule. But this is impossible since it is the collapse rule that is needed for explosion in the first place. In other words, there is no way to get from $\bot$ to A more directly than via the collapse rule. Thus, there is no reduction for the "detour" $\mathcal{D}_\bot$:

$$\cfrac{\cfrac{\begin{matrix}\mathcal{D}\\ \bot\end{matrix}}{A \vee \bot}\,\vee\mathrm{I}_r}{A}\,\text{collapse} \qquad \not\xRightarrow{\text{reduces to}}$$

Recall, however, that we were able to find a reduction for the case when $A \vee \bot$ was constructed from A. What does that mean? It means that the collapse rule behaves as a locally sound elimination rule for detours where $A \vee \bot$ was derived from A but it behaves as a locally unsound elimination rule for detours where $A \vee \bot$ was derived from $\bot$. To put it differently, in the presence of $\bot$, the collapse rule becomes too strong as a disjunction elimination rule: it allows us to derive more than the disjunction introduction rule at hand permitted. Namely, it allows us to derive from $\bot$ any A, not just $\bot$ again.

But notice that this is exactly what should have been expected: we need the collapse rule to be stronger in this particular case, otherwise, it would not be able to capture the explosion principle! In other words, the collapse rule as an elimination rule has to allow us to derive more than the corresponding disjunction introduction rules permit or else we would not be able to get from $\bot$ to any A.

To sum up, the collapse rule behaves as harmonious, i.e., it is both locally sound and locally complete, in detours where $A \vee \bot$ was derived from A. However, in detours where $A \vee \bot$ was derived from $\bot$, the collapse rule loses local soundness, and thus also harmony (however, it retains local completeness). This is, however, not a defect that should be viewed with suspicion. On the contrary, it is what allows us to capture the explosion principle: not only does the collapse rule allow us to derive *more* than disjunction introduction rules permit, it allows us to derive *anything*, just as a proper explosion principle should. And recall that our main goal is not to invalidate EFQ or the explosion principle it captures but to offer a new alternative justification for it. In other words, having a collapse rule that is harmonious even in the presence of $\bot$ would be detrimental – we would not be able to use it to derive EFQ.

The interesting upshot of this is that the explosion principle ceases to be a stipulated property of $\bot$ (as in the case of EFQ, further relying on the semantic principle of vacuous truth for justification) and becomes a proof-theoretic consequence of the interaction between disjunction and absurdity, specifically, between the disjunction introduction rule applied to $\bot$ and the following application of the collapse rule. This leads to an interesting new perspective on the explosion principle: it is a form of local unsoundness of the collapse rule in the presence of $\bot$.

So, can we justify the collapse rule with respect to the disjunction introduction rules? In other words, is the collapse rule harmonious? As we have seen above, the answer to this question is not straightforward: in some cases, it is harmonious and in some cases, it is not and it is the lack of harmony in the latter cases that is actually needed for the collapse rule to function as intended, i.e., to be able to produce the explosion principle. And it is important to emphasize that during this process we do not presuppose it.[25] As discussed above, the explosion principle is really just a consequence of the fact that the collapse rule is locally unsound in the presence of $\bot$, i.e., that there is no reduction for a detour formed by an application of $\vee\mathrm{I}_r$ followed by an application of the collapse rule

[25]This concern was brought up by one of the reviewers.

as discussed above. What we do presuppose, however, is the implosion principle captured by the collapse rule which, with the help of $\vee\mathrm{I}_r$, then allows us to derive any A from $\perp$ as shown in derivation $\mathcal{D}_\perp$.

Remark 3. One of the reviewers inquired about the relation between harmony in the sense adopted here and normalization, i.e., the process of reducing a proof into a normal form by removing all the unnecessary detours (see, e.g., Negri and von Plato (2001)). To properly address this topic would require more space than is available, however, we give at least a brief overview. First, as normalization is concerned with the removal of unnecessary detours, it is specifically related to the property of local soundness, not harmony as such. In other words, proofs constructed via rules that are locally sound should be also normalizable. Second, as we have seen, the collapse rule gives rise to two kinds of detours: one that is removable, i.e., a detour of kind $\mathcal{D}_A$, and one that is not removable (and should not be), i.e., a detour of kind $\mathcal{D}_\perp$. From this perspective, the simple answer to the question if every proof involving the collapse rule can be reduced to a normal form by removing all the detours has to be negative.[26]

5 Disjunctive syllogism and interderivability

It is known that EFQ can be derived and thus justified by the rule of disjunctive syllogism (DS), also known as modus tollendo ponens:

$$\frac{A \vee B \qquad \neg B}{A}$$

The derivation proceeds as follows:[27]

$$\dfrac{\dfrac{\perp}{A \vee \perp}\,\vee\mathrm{I}_r \qquad \dfrac{[\perp]^1}{\neg\perp}\,{\to}\mathrm{I}_1}{A}\,\mathrm{DS}$$

However, the other way holds as well, i.e., DS can be derived using EFQ. Specifically, DS can be established via the following derivation:

$$\dfrac{A \vee B \qquad [A]^1 \qquad \dfrac{\dfrac{\neg B \qquad [B]^2}{\perp}\,{\to}\mathrm{E}}{A}\,\mathrm{EFQ}}{A}\,\vee\mathrm{E}_{1,2}$$

[26] Note, however, that this reply presupposes that detours such as $\mathcal{D}_\perp$ are detours just like any other, which can be contested. For example, given the fact that the collapse rule says nothing about the general case $A \vee B$ and pertains specifically to disjunctions of the form $A \vee \perp$, it can be argued that its immediate application after the disjunction introduction rule applied to $\perp$ does not constitute a proper detour but rather a pseudo-detour which need not be removed during normalization.

[27] See, e.g., von Plato (2014), p. 46.

It follows that whatever can be proved with EFQ can also be proved with DS and vice versa since they are interderivable. In other words, they have the same deductive strength.

However, despite this fact, it is almost always EFQ that is chosen to be the more basic rule. Of course, there are good reasons for that, both theoretical and practical. For example, EFQ can fit nicely inside the introduction and elimination rules paradigm, which DS cannot. Furthermore, EFQ makes use of only a single logical connective, i.e., $\bot$ (it is a pure rule in Dummett's terminology, Dummett (1991), p. 257), while DS makes use of three connectives: $\vee$, $\to$ and $\bot$.[28] In short, EFQ seems like a more basic rule than DS and for this reason, DS is typically understood as justified via EFQ, not the other way around.

Now, we already know that EFQ can be derived by the collapse rule, but how does it stand in relation to DS? Can we also derive DS using the collapse rule? As it turns out, the answer is yes. The derivation proceeds as follows:

$$\cfrac{A \vee B \qquad [A]^1 \qquad \cfrac{\cfrac{\cfrac{\neg B \qquad [B]^2}{\bot}\,{\to}\mathrm{E}}{A \vee \bot}\,\vee\mathrm{I}_r}{A}\,\text{collapse}}{A}\,\vee\mathrm{E}_{1,2}$$

Thus, the collapse rule can be used to derive and thus justify both EFQ and DS. But can we also derive the collapse rule by EFQ and DS? In other words, can we show that all these three rules have the same deductive strength? We already know the answer for EFQ, i.e., the collapse rule can be derived via EFQ as follows (for convenience we repeat the derivation from Section 4.1):

$$\cfrac{A \vee \bot \qquad [A]^1 \qquad \cfrac{[\bot]^2}{A}\,\mathrm{EFQ}}{A}\,\vee\mathrm{E}_{1,2}$$

Now, all that is left is to show how we can establish it via DS. The derivation proceeds as follows:

$$\cfrac{A \vee \bot \qquad \cfrac{[\bot]^1}{\neg\bot}\,{\to}\mathrm{I}_1}{A}\,\mathrm{DS}$$

Thus, we can conclude that aside from EFQ and DS, there is another potentially basic rule that is just as deductively strong as they are and that can be used to justify them: it is the collapse rule. And it brings yet another informal justification distinct from those of EFQ and DS: it stands on the fact that a choice between A and no choice is not really a choice.

Remark 4. Someone might ask why we bother with the collapse rule if there is already a perfectly good justification of EFQ via the DS. There are three main reasons:

[28] Recall that we define negation $\neg A$ as $A \to \bot$.

i) philosophical: the collapse rule is conceptually different than DS as it makes use of the notion of absurdity instead of contradiction.[29] Furthermore, many systems, most notably intuitionistic logic, take absurdity as a primitive notion of the system and use it to define negation, which is then understood as an implication of absurdity. From this perspective, the collapse rule is also simpler than DS as it does not invoke the notion of implication.[30]

ii) technical: the collapse rule can be treated relatively straightforwardly as an elimination rule for disjunction, which then opens it to the possibility of justification in terms of harmony, specifically in terms of its local soundness and local completeness.

iii) explanatory: the explanations of the validity of EFQ are often related or reduced to the explanations of the validity of DS (or vice versa).[31] But now, we have a new option: both explanations can be reduced to the explanations of the validity of the collapse rule.

Note, however, that point iii) currently extends only to explosive logics such as intuitionistic and classical logic. In non-explosive logics, such as relevant logic and paraconsistent logic, the issue is more complicated. For example, in Tennant's intuitionistic relevant system of Core Logic (Tennant (2017)), we can derive the statement (where "$\vdash$" denotes the deducibility relation):

$$B \wedge \neg B \vdash A \vee (B \wedge \neg B)$$

which essentially corresponds to an instance of the disjunction introduction rule applied to $\bot$.[32] Furthermore, we can also derive the statement:

$$A \vee (B \wedge \neg B) \vdash A$$

which corresponds to an instance of the collapse rule. However, we cannot put these two statements together via cut and derive:

$$B \wedge \neg B \vdash A$$

which corresponds to the explosion principle, as it would require the use of unrestricted transitivity of deduction, which Tennant's system rejects (see Tennant (2017), p. 47). In other words, Core Logic's variant of our derivation:

[29] The relationship between these two notions is, however, far from simple. Many authors do not distinguish between contradiction and absurdity at all (see, e.g., van Dalen (2013), p. 30, Mancosu et al. (2021), p. 37), some of them even treat them as "primitive (unexplained) notion" (see, e.g., Troelstra and van Dalen (1988), p. 9, Heyting (1956), p. 98), while others distinguish between them (see, e.g., Schroeder-Heister (2012), p. 79, Smith (2020), p. 146). This is, however, an issue outside the scope of the present paper.

[30] Of course, those systems that regard negation as a primitive notion would view the issue differently.

[31] Recently, see, e.g., Hewitt (2022), pp. 280–281, Smith (2020), pp. 198–202.

[32] We have to replace absurdity $\bot$ with contradiction $B \wedge \neg B$ as Tennant's system does not regard $\bot$ as a proposition.

$$\dfrac{\dfrac{\perp}{A \vee \perp}\,\vee\mathrm{I}_r}{A}\,\text{collapse}$$

establishing EFQ as a derived rule via the collapse rule will simply not work.[33]

6 Conclusion

Despite EFQ and DS having the same deductive power, it is typically EFQ that is considered the more basic one. However, EFQ can be difficult to justify on its own: why should indeed everything follow from absurdity? In this paper, we have proposed a new justification for the EFQ based on a new rule we call the collapse rule. With this rule, we can treat EFQ as a derived rule, just like DS. Thus, to the dilemma of whether EFQ or DS should be taken as the more basic rule in explosive logics, we add a third option: neither, as we can find another rule capable of deriving them both.

The collapse, embodying the principle of implosion, is a hybrid rule between EFQ and DS: similarly to EFQ, it has only a single premise and it makes use of the meaning of $\perp$ (specifically, that it is a proposition that cannot be true) and, similarly to DS, it is an impure rule[34] and it makes use of the meaning of $\vee$ (specifically, that it is a proposition that is true when one of its disjuncts is true). Furthermore, assuming that $A \vee \perp$ has been derived from $\perp$, the collapse rule can lead to trivial proofs of any A just like EFQ. And just like DS, the collapse rule can be viewed as a disjunction decision rule:[35] just from the information contained in the premise(s), we can decide which disjunct holds.

We have provided two types of justifications for the collapse rule. Specifically, we have offered two informal justifications: one is based on the idea that the collapse rule is a rule of immediate inference, and the other is based on the idea that disjunction can be interpreted as an incomplete communication. The formal justification is based on the idea that the collapse rule is an elimination rule for disjunction (which is in harmony with disjunction introduction rules as long as the premise of the collapse rule was derived from A and not from $\perp$). The collapse rule also comes with its own intuitive explanation: we can regard it as capturing the idea that a choice where only one option can be made true can be reduced to that very option. In other words, the collapse rule can be understood as a rule for simplifying choices of the form $A \vee \perp$ to just A. Of course, more could be said about the philosophical background of the collapse rule or its implementation into other kinds of logical frameworks such as, e.g., propositions-as-types logic or sequent calculus. However, since the main goal of this paper is to show how we can logically justify EFQ and DS using the

[33] We thank one of the reviewers for this important observation.

[34] Note that this rule can never be pure: we do not have the option to regard $\perp$ as a structural punctuation mark here, as it appears as a part of a proposition, hence it has to be understood as a logical connective.

[35] Or alternatively, as a communication completion rule for communications of the form $A \vee \perp$.

collapse rule in the setting of natural deduction, not to directly explore the rule itself, we leave these topics for future work.

The main advantage of the collapse rule is that it effectively sidesteps most of the criticism laid on EFQ by treating it as a derived rule instead of a basic rule. And while it can be tricky to explain why everything should follow from absurdity, it seems quite unproblematic to accept the principle that a choice with only one feasible option can be simplified just to that option. The main disadvantage is that the collapse rule does not fit as nicely within the introduction and elimination rules paradigm as EFQ. Also, the loss of local soundness in case the premise for the collapse rule was derived from $\perp$ can be regarded as a drawback by some, however, justified it might be.

To conclude, aside from EFQ and DS, there is another potentially basic rule that is just as deductively strong as they are and that can be used to justify them: it is the collapse rule, which comes along with yet another informal justification distinct from those of EFQ and DS. Simply put, it captures the idea that a choice between A and no choice is not really a choice at all.

Declarations

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