Absurdity as the impossible command in natural deduction

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Abstract

In this paper, we propose a new approach to absurdity in the context of natural deduction for intuitionistic and classical logic. It combines aspects of both the logical approach, which treats absurdity as a propositional constant, and the structural approach, which treats absurdity as a structural punctuation mark signaling the dead end of derivations. In particular, we will treat absurdity as an impossible command, i.e., a speech act composed of an imperative force indicator, and the False propositional constant, i.e., a proposition that cannot be true by definition. In return, we obtain a framework that constitutes a middle ground between the logical and structural approaches. For example, it allows us to consider the ex falso quodlibet rule as a kind of structural rule, specifically, a semi-structural rule, and at the same time also maintain that negation can be reduced to implication of absurdity, specifically, to its propositional content.

keywords: natural deduction, proof theory, absurdity, falsity, speech acts

1 Introduction

The notion of absurdity commonly denoted by $\perp$ makes four key appearances in the rules of natural deduction for intuitionistic and classical logic.[1] Let us start with intuitionistic logic. The first appearance is in the Ex Falso Quodlibet rule (EFQ), also known as the principle of explosion or the absurdity rule:

$$\frac{\perp}{A}\,\text{EFQ}$$

which we can read as "If we can derive absurdity, then anything follows".

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[1]There are, of course, many other important appearances of $\perp$ in other logical systems, but to keep the present investigation manageable within a single paper, we will focus only on these two most standard logical systems.

The second and third appearances are in the negation introduction rule (sometimes also known as the Intuitionistic Reductio ad Absurdum rule[2]) and its corresponding negation elimination rule (sometimes also referred to as the Ex Contradictione Falsum rule):

$$\frac{\begin{matrix}[A]\\ \bot\end{matrix}}{\neg A}\;\neg\mathrm{I} \qquad\qquad \frac{\neg A \qquad A}{\bot}\;\neg\mathrm{E}$$

We can read these rules as "If absurdity follows from the assumption A, then we can derive $\neg A$" and "If both $\neg A$ and A can be derived, then absurdity follows". Note that if we define negation via implication (as is common in intuitionistic logic):

$$\neg A =_{df} A \to \bot$$

i.e., if we understand the negation of A as reducible to the implication of absurdity, we can consider the ¬I and ¬E rules displayed above just as special cases of implication introduction rule and implication elimination rule (modus ponens) when $B = \bot$:

$$\frac{\begin{matrix}[A]\\ B\end{matrix}}{A \to B}\;\to\mathrm{I} \qquad\qquad \frac{A \to B \qquad A}{\bot}\;\to\mathrm{E}$$

Now, the three appearances of $\bot$ in EFQ, ¬I, and ¬E mentioned above also apply to classical logic. The fourth appearance of $\bot$, is, however, specific to classical logic and it appears in the Reductio ad Absurdum rule (RAA), also known as the rule for indirect proofs:

$$\frac{\begin{matrix}[\neg A]\\ \bot\end{matrix}}{A}\;\mathrm{RAA}$$

which we can read as "If we can derive absurdity from the assumption $\neg A$, then A follows" (note its similarity to the negation introduction rule). If we add this rule to intuitionistic logic, we obtain classical logic. Furthermore, note that this rule is essentially a generalized variant of EFQ: it is EFQ that allows us to withdraw assumptions. Or to put it differently, EFQ is a special case of RAA (see, e.g., Prawitz (1965), p. 21).

Let us refer to all these rules involving absurdity, i.e., EFQ, ¬I, ¬E, and RAA, simply as *absurdity rules*. As evidenced by them, $\bot$ does a lot of heavy lifting in natural deduction for intuitionistic and classical logic. Namely, it is used for capturing the principle of deductive explosion, for introducing and eliminating the negation connective, and for formalizing the indirect proof technique.

[2] Note, however, that this terminology is not entirely appropriate as the negation introduction rule is not characteristic of intuitionistic logic since it already appears as a rule of minimal logic. Thus, it can be misleading to call it Intuitionistic Reductio ad Absurdum.

But despite the undeniable significance of $\perp$, not all questions regarding its nature are settled. For example, there are two main opposing views one can take in regards to the status of absurdity $\perp$ in the context of natural deduction for both intuitionistic and classical logic:

(1) $\perp$ is a *logical* notion: most commonly treated as a propositional constant – something with meaning that should be conveyed by an appropriate meaning-giving introduction and/or elimination rules or directly stipulated (see, e.g., Gentzen (1969); Prawitz (1965); Martin-Löf (1971); Troelstra and van Dalen (1988); Troelstra and Schwichtenberg (2000); Negri and von Plato (2001); von Plato (2014); Smith (2020)), and

(2) $\perp$ is a *structural* notion: a so-called structural punctuation mark – something without meaning on its own only marking certain structural properties of proofs, specifically, reaching a dead end (see, e.g., Tennant (1999); Rumfitt (2000); Steinberger (2009); Incurvati and Schlöder (2019); Murzi (2020)).

(2) grew out of dissatisfaction with the account of $\perp$ provided by (1) (see Tennant (1999), see also Sanz (2004)) and slowly became a commonly accepted alternative to it, although (1) still remains the prevalent view in both textbooks and research papers.

These two approaches influence how we interpret the rules in which $\perp$ appears. For example, let us consider again the EFQ rule:

$$\frac{\perp}{A}\,\text{EFQ}$$

On the first view (1), it becomes the elimination rule for $\perp$ (see, e.g., Martin-Löf (1971), p. 189, Prawitz (1973), p. 243). On the second view (2), however, it becomes a structural rule (see, e.g., Steinberger (2009)).

Depending on which view we take, different requirements are then asked of EFQ. If we regard EFQ as an elimination rule, then we should be able to justify it in regards to its introduction rule or its lack of.[3] On the other hand, if we regard EFQ as a structural rule exhibiting the assumed properties of the underlying notion of logical consequence, no such justification is needed. Of course, we should still be able to provide some sort of justification for why we adopt this structural rule but the point is that it will not be justified in terms of the relation between introduction and elimination rules.

The choice between (1) and (2) also affects the proof-theoretic properties of certain rules. For example, the classical Reductio ad Absurdum rule (RAA):

$$\frac{\begin{array}{c}[\neg A]^i\\ \vdots \\ \perp\end{array}}{A}\,\text{RAA}_i$$

[3]We assume here the verificationist view according to which elimination rules should be justified with respect to introduction rules, which are regarded as the primary meaning-giving rules (see, e.g., Gentzen (1935), Dummett (1991), Prawitz (2006), Martin-Löf (1996)).

is an impure rule[4] on the first approach (1) as it makes use of two logical connectives, i.e., $\neg$ and $\perp$, but becomes a pure rule on the second reading (2) as $\perp$ is no longer regarded as a logical connective. This has important consequences from the inferentialist perspective where purity is often seen as a criterion for rules to be meaning-giving.

Persuasive arguments have been provided for both of these approaches and our goal is not to reevaluate them or to expand them. Instead, in this paper we want to explore a hybrid approach to absurdity that combines aspects of both the logical and the structural approach. On this hybrid approach, absurdity will not be regarded as a logical constant as in (1), or a structural punctuation mark as in (2), but as a speech act, i.e., a compound [*force*] + [*proposition*] composed of a force indicator and a propositional content, that exhibits both logical and structural features. In particular, we will regard absurdity as a compound of imperative force indicator and the False propositional constant and interpret it as the impossible command.

Structure of the paper. In Section 2, we introduce our hybrid approach to absurdity that is based on the idea of treating it as the impossible command. We also briefly present the logical and structural approaches to absurdity and discuss why we want to associate imperative force with absurdity. In Section 3, we will show how to implement this notion of absurdity into a natural deduction system and propose alternative interpretations of the relevant absurdity rules.

2 Hybrid approach

Before introducing our hybrid approach to absurdity, let us briefly survey the logical (1) and structural (2) approaches mentioned in the previous section.

On the view (1), $\perp$ is understood as a propositional constant. Formally, this typically means treating it either as a zero-place connective or as an atomic proposition with definite meaning.[5] Either way, it is a part of the object language and can be used as a constituent of propositions (thus, it allows us to define negation as $A \to \perp$). If we choose the first route, its meaning is to be specified via some introduction and/or elimination rules just like for any other connective. There are two main alternatives: $\perp$ as having no introduction rule and only elimination rule (= EFQ) (see, e.g., Martin-Löf (1971), p. 189), which is by far the most common one, and $\perp$ as having an infinitary introduction rule:

$$\frac{p \qquad q \qquad r \qquad \dots}{\perp}\,\perp\mathrm{I}$$

where $p, q, r, \dots$ are propositional variables and corresponding elimination rule (= EFQ) (see Dummett (1991), p. 295).

[4] "A rule may be called 'pure' if only one logical constant figures in it" (Dummett (1991), p. 257).

[5] It is worth noting that although treating $\perp$ as an atomic proposition is quite common (see, e.g., Mancosu et al. (2021), p. 37.), it is not always the case. For example, sometimes it is considered as a prime proposition, while prime propositions are atomic propositions and $\perp$. See, e.g. Troelstra and Schwichtenberg (2000), p. 2.

If we choose the second route, $\bot$ gets its meaning from outside the system of inference rules as is typical for atomic propositions in proof theory, i.e., the understanding of $\bot$ is presupposed to be given. In other words, it is effectively treated as a primitive, further unanalysable notion. For example, Gentzen called $\bot$ the "definite proposition" Gentzen (1969) (p. 70), particularly "the false proposition", i.e., a proposition that cannot be true by definition.[6]

On the view (2), $\bot$ has a seemingly unique role. It is not a logical connective, not even an atomic proposition but a structural/logical punctuation mark[7] that informs us that a derivation has reached a "logical dead end" (Tennant (1999), p. 205).[8] As Tennant (1999) further explains:

> [A]n occurrence of $\bot$ is appropriate only within a proof [...], as a kind of structural punctuation mark. It tells us where a story being spun out gets tied up in a particular kind of knot – the knot of patent absurdity ... (Tennant (1999), p. 204)

So, the intended effect of a rule such as negation elimination ¬E:

$$\frac{\neg A \qquad A}{\bot}\,\neg\mathrm{E}$$

is that it ends a derivation:

$$\frac{\neg A \qquad A}{\boxed{\text{dead end}}}$$

In other words, once we reach both $\neg A$ and A, there is no way forward.[9] Thus, $\bot$ is effectively regarded as a purely structural dead-end sign that has no logical content. This in turn allows us to turn some logical rules into structural ones (namely, EFQ) and some impure rules into pure ones (namely, RAA). Both of these prospects are especially appealing from the inferentialist perspective as they open new possibilities for the justification of these rules.

Both approaches (1) and (2) have their strengths and weaknesses but arguably one of the most distinctive and useful features of the logical approach is that it allows us to define negation via implication of absurdity (which is a

[6] See also, e.g., Heyting (1956), p. 98, Troelstra and van Dalen (1988), p. 2, Martin-Löf (1996), Hand (1999), Kürbis (2019).

[7] In this respect, Tennant (1999)'s approach is distinct from Došen (1989)'s that treats all logical constants as punctuation marks, not just $\bot$.

[8] See also Rumfitt (2000) or more recently Incurvati and Schlöder (2019), Murzi (2020).

[9] Some might consider the interpretation of $\bot$ as a punctuation mark for logical dead ends to be incompatible with the EFQ rule. After all, doesn't this rule specifically tell us that $\bot$ is not the end of deduction, but the beginning of any possible deduction? It is true that originally Tennant's structural approach to absurdity led to a relevant, i.e., non-explosive logic, however, it is not tied to it. For example, EFQ can be regarded as a structural rule that tells us something about the assumed properties of the underlying relation of logical consequence (see, e.g., Steinberger (2009)) – in this particular case, that it is explosive. And to begin a new deduction via explosion, some other deduction had to end first, which is precisely the point of EFQ under this interpretation.

common practice basically in all systems of intuitionistic logic and many others). On the other hand, one of the most distinctive and useful features of the structural approach is arguably that it allows us to treat EFQ as a structural rule (so that we do need to position EFQ within the framework of introduction and elimination rules, and thus sidestep the whole discussion about the meaning specification of absurdity understood as a logical connective/propositional constant) and regard RAA as a pure rule (so that we can regard it as better behaved from the inferential meaning perspective).

Our goal is to propose a hybrid approach that combines these desirable features of both approaches (1) and (2). Specifically, our approach will allow us to define negation via implication and absurdity, particularly, via implication and absurdity's propositional content, similarly to the logical approach (1), and simultaneously it will allow us to treat EFQ as a kind of structural rule, particularly, as a semi-structural rule (to be specified in Section 3), similarly to the structural approach (2). However, unlike in the structural approach, in our approach, RAA will not be regarded as a pure rule.

The underlying idea is to treat absurdity $\perp$ not as a purely logical notion in the sense of (1) or a purely structural notion in the sense of (2) but as a combination of the two giving rise to a hybrid notion of absurdity. More precisely, we will treat absurdity as the [*imperative force*] + [*false proposition*] compound, i.e., a speech act composed of an imperative force indicator (representing the structural aspect) and the False propositional constant (representing the logical aspect, particularly, the propositional content of absurdity), which we will interpret as the impossible command.

Now, with this in mind, our task will be three-fold: in this section, we will (i) explain why we want to associate imperative force with absurdity, (ii) and what exactly we mean by the impossible command, and in the next section, we will (iii) show how to implement this notion of absurdity into a natural deduction system.

2.1 Imperative force

Let us begin with a small detour and talk about the distinction between propositions and assertions. It is not uncommon to interpret the rules of natural deduction systems as operating on assertions of the form:

$$\vdash A$$

i.e., on compounds composed of an assertoric force indicator $\vdash$ and a proposition A, and not just on propositions alone. See, e.g., Negri and von Plato (2001):[10]

> Rules of inference are of the form: "If it is the case that A and B, then it is the case that C." Thus they do not act on propositions but on assertions.

[10] We will prefer the term "force" in this paper and treat it as analogous with mood. Generally speaking, however, there are good reasons to keep these two notions separate. See, e.g., Ranta (1994), p. 26, Stenius (1967), Hare (1970).

> We obtain an assertion from a proposition A by adding something to it, namely, an assertive mood [= force] such as "it is the case that A." Frege used the assertion sign $\vdash A$ to indicate this but usually the distinction between propositions and assertions is left implicit. Rules seemingly move from given propositions to new ones. (Negri and von Plato (2001), p. 3)

This means that the properly disclosed forms of natural deduction rules operate on [*force*] + [*proposition*] compounds. For example, the rules involving absurdity should be depicted as follows (here we presuppose that $\bot$ is a logical notion in the sense of (1) since it makes no sense to associate force indicator with a structural punctuation mark):[11]

$$\dfrac{\vdash \bot}{\vdash A}\,\mathrm{EFQ} \qquad \dfrac{\vdash \neg A \qquad \vdash A}{\vdash \bot}\,\neg\mathrm{E} \qquad \begin{array}{c}[A]\\ \dfrac{\vdash \bot}{\vdash \neg A}\end{array}\,\neg\mathrm{I} \qquad \begin{array}{c}[\neg A]\\ \dfrac{\vdash \bot}{\vdash A}\end{array}\,\mathrm{RAA}$$

And since it is generally assumed that all the standard rules of natural deduction, including those above, operate only with assertions (or hypothetical assertions), the force indicators are typically omitted. As von Plato (2014) puts it:

> The sentences that we utter come with a mood [= force] that is usually understood by the listener. We do not need to add in front of every sentence *it is the case that...*, even if we sometimes do it for emphasis or clarity. (von Plato (2014), p. 8)

But what if not all force indicators are assertoric? What if some of the forces used in natural deduction are not "understood by the listener", as von Plato (2014) said? Particularly, what if the force associated with absurdity is not assertoric but of a different kind?

Now, what reasons do we have to challenge the common assertoric view of absurdity? The key indication that associating assertoric force with absurdity might not be sufficient comes from examining the negation elimination rule $\neg$E which is the only absurdity rule that actually allows us to derive $\bot$ as the conclusion. First, recall that according to the structural approach (2), $\bot$ in the conclusion of the $\neg$E rule should be understood as a sign marking dead ends of derivations (not that since $\bot$ is understood structurally, i.e., it has no propositional content, no force indicator is to be attached):

$$\dfrac{\vdash \neg A \qquad \vdash A}{\bot}\,\neg\mathrm{E}$$

[11]The vertical arrangement of the form $\begin{array}{c}A\\ \vdash B\end{array}$ represents a speech act of hypothetical assertion, i.e., an assertion B made under certain assumption A (Martin-Löf (1984)). In other words, there is no assumptory force indicator associated with A as assumption is not regarded as a sui generis speech act but rather as an "assumption" part of the speech act of hypothetical assertion. For justification of this position, see, e.g., Frege's *Über die Grundlagen der Geometrie* (1906) in Frege (1967), p. 425, Tichý (1988), chap. 13, or Pezlar (2014, 2013). For the issues we can encounter when we view assumptions as standalone speech acts, see Kürbis (2023).

Note, however, that if we were to stick with Tennant's original "traffic signs" metaphor, $\perp$ should be perhaps even more suitably understood not as a dead-end sign but rather as a logical stop sign:

$$\frac{\vdash \neg A \qquad \vdash A}{\boxed{\text{stop}}}$$

after all, its purpose is not really just to *inform us* that absurdity has been revealed but rather to *make us* stop at first sight of it. In other words, $\perp$ gives us not only information but also a directive on what to do, more specifically, on what not to do, i.e., not to continue the derivation. These considerations were directly motivated by the fact that Tennant was interested in relevant, i.e., non-explosive logic, which also means no EFQ rule, i.e., no continuation after $\perp$ has been reached (unless we discharge the assumption that led to it and derive its negation instead, i.e., apply negation introduction $\neg$I).

The idea that $\perp$ not only informs us but also directs us towards some specific action, namely stopping the derivation, can be, however, found in the logical approach (1) as well. Consider the following: Typically we want our formal systems, as well as our theories and beliefs, to be free of absurdities. So, when an absurdity appears, it is an important discovery that requires immediate attention: it makes us stop so that we can carefully consider its source and the ways how to possibly avoid it by revising our assumptions.[12] This stopping power seems to be an aspect of absurdity that is not associated with any other logical connective, propositional constant, or assertion thereof.

We believe it is this aspect of absurdity, i.e., its inherent ability to stop the derivations to give us an opportunity to revise our assumptions, that is much easier and natural to explain if we regard absurdity as associated not only with informing our reasoning but also with directing it. In other words, absurdity is not only about conveying information (which we typically expect from assertions) but also about requiring some action, something to be done (which we typically do not expect from assertions), namely stopping the derivation so that we can revise our assumptions. And as we will argue below, one sure way how to stop somebody's progress in a derivation is to give them a task that cannot be completed. So, considering this directive aspect of absurdity, it seems reasonable to associate an imperative force with it as the assertoric one does not seem sufficient anymore.[13] However, we want to emphasize that we do not want to claim that there is anything fundamentally flawed with associating assertoric force with absurdity, only that the imperative force seems to make it easier to explain the source of its derivation-stopping power. In short, there seems to be

[12]This might also help to explain why the force associated with absurdity might not be understood by the listener: our tendency to stop at discovering absurdities comes so naturally to us we do not pay any special attention to it.

[13]The idea that absurdity is associated with the imperative force and the ability to stop derivations can also find some justification in various Wittgenstein's remarks on the topics of reductio ad absurdum proof technique and contradictions (see, e.g., Wittgenstein (1978), III-58, 130e; V-29, 182; Wittgenstein (1975), p. 138, p. 179, p. 185, p. 223). Properly investigating these connections would be, however, a task better suited for a dedicated paper.

no straightforward way to explain the directive role of absurdity if we assume its nature is purely informative.

2.2 Impossible command

Now, if there is a force associated with $\perp$ it can no longer be considered as a purely structural notion, it has to be regarded as a [*force*] + [*proposition*] compound, i.e., a speech act. We already know the type of force we want to invoke, i.e., imperative one, but what proposition should we use to construct a command that would stop a derivation?

There are possibly other options but one certain way to stop somebody from progressing further in a derivation is to simply give them a proposition to prove that cannot be proved, i.e., give them a task that cannot be completed. This will ensure that they will get stuck, or to borrow Tennant's term, "tied up in a particular kind of knot", and will never be able to continue in the derivation further.

And how can we make sure that the proposition we give them to prove cannot be proved? This is straightforward: we just give them a proposition that cannot be proved by definition. And this is exactly the most common interpretation of $\perp$ in the logical approach (1), i.e., $\perp$ as something that can never be true.

Now, we have both the force and the propositional content, so we can explain how to express absurdity in terms of a [*force*] + [*proposition*] compound. First, we will need symbols for the imperative force and for the False propositional constant. For the imperative force indicator, let us re-purpose the symbol $\perp$ since it goes well with the symbol $\vdash$ that is used as the assertoric force indicator.[14] Thus, just as we can make an assertion by taking a proposition A and adding to it an assertoric force $\vdash$ (see, e.g., Negri and von Plato (2001)), we can make a command by taking a proposition A and adding to it an imperative force $\perp$. For example,

$$\perp A$$

is a command (order, imperative, ...) to do A, or, perhaps more fittingly, it is a "command that $[A]$ be made true" (Dummett (1973), p. 307), i.e., it gives us the obligation to make A true. If A is made true, we can say that the command is obeyed or fulfilled, otherwise, it is disobeyed or unfulfilled.

For the False propositional constant, i.e., a proposition that cannot be proven by definition, let us introduce a new symbol $\mathcal{F}$. Other possible interpretations of $\mathcal{F}$ are also falsity, falsehood, or the false. Similarly to $\perp$ in the traditional approach, $\mathcal{F}$ can be a constituent of other propositions, i.e., it can be used to define negation as $A \to \mathcal{F}$, etc. Alternatively, we could also understand $\mathcal{F}$ as the vacuous proposition, i.e., the proposition that has by definition no truthmaker, whatever that might be (constructive proofs, classical proofs, state

[14] It is worth noting that Humberstone (2011) (pp. 1182–1183) also briefly considers $\perp$ as a force indicator, however, not as a unary one as we do but as a nullary one and without any specific force associated with it. What happens when we consider $\perp$ not as a unary force indicator but as a nullary one is further explored in Pezlar (forthcoming).

of affairs, facts, ...), which, consequently, makes it false by default. In the end, $\mathcal{F}$ shouldn't be any less or more mysterious than, e.g., the empty set in set theory, or 0 in arithmetic, to borrow an example from Tennant (1999).

Now, putting together $\perp$ and $\mathcal{F}$, we obtain the [*force*] + [*proposition*] compound

$$\perp \mathcal{F}$$

How do we interpret it? It is a command to do $\mathcal{F}$, or more precisely, to make $\mathcal{F}$ true. And since $\mathcal{F}$ cannot be true by definition (i.e., it is the False propositional constant), $\perp \mathcal{F}$ then effectively becomes an impossible command, i.e., a command that cannot be obeyed ("Prove the unprovable!"). And, as argued above, the most natural effect of such command is nothing: no action at all, i.e., it is effectively stopping the derivation.

Remark. Extending logic with commands is, of course, nothing new and has been considered in the past (Menger (1934), Jörgensen (1937), Belnap (1990)) and by many since then. At this point, we are, however, not concerned with developing an imperative logic, rather we just want to explore the idea that absurdity is the impossible command. Also, for us, a command is not a propositional operator, as is common in imperative logics, but a speech act with an imperative force, thus, e.g., $A \to \perp \mathcal{F}$ would be senseless in our approach – we agree with Dummett (1973) (p. 303) that the imperative force "governs the sentence as a whole, and not its constituent clauses taken separately."[15]

This line of reasoning, i.e., that $\perp \mathcal{F}$ should result in inaction is also supported by the alternative interpretation of $\mathcal{F}$ as the vacuous proposition. Consider, e.g., a command of the form $\perp A \wedge B$. The propositional part $A \wedge B$ can be understood as the content of the command. In this case, the command to make true $A \wedge B$ is fulfilled (obeyed) when both A and B are made true. The propositional part tells us what we should do if we want to obey the command – it gives us instructions. Now, let us return to the command $\perp \mathcal{F}$. Since $\mathcal{F}$ can be understood as the vacuous proposition, it gives us no instructions. Thus it is effectively a vacuous command that tells us to do nothing. Or rather, less than nothing, because even doing nothing is something. It stops the derivation at hand since we do not know what to do when we receive the command to do "nothing".[16] And it is this impossible command $\perp \mathcal{F}$ that, we believe, is best understood as absurdity.[17]

Thus, the negation elimination rule should look as follows:

[15] Note that by considering such a framework, we are essentially assuming a bilateral framework but instead of the standard duality of assertions and denials, we work with assertions and commands, specifically, with the impossible command. We thank Hitoshi Omori for this observation.

[16] Recall that $\mathcal{F}$ has no introduction rules (= no instructions on how to canonically prove it), since it is false by stipulation.

[17] One of the reviewers asked if we could choose to express absurdity, e.g., as $\perp 0 = 1$. We could but this definition would make it context-dependent as it relies on arithmetical (extralogical) notions. We are interested in a more general approach to absurdity that is not tied to a specific context.

$$\frac{\vdash \neg A \qquad \vdash A}{\perp \mathcal{F}}$$

It essentially says that an application of the negation elimination rule leads to the unfulfillable obligation to prove the False propositional constant, which then results in stopping the derivation.

Remark. Some might wonder if we really need the notion of the impossible command $\perp \mathcal{F}$ for this kind of explanation of the negation elimination rule. Wouldn't the notion of the impossible assertion $\vdash \mathcal{F}$ do just as well? Yes, it could, but only if we were to adopt a position where even assertions give rise to obligations, not just commands (see, e.g., Martin-Löf (2020)). Note, however, that even with this concession, $\vdash \mathcal{F}$ would describe a different notion of absurdity from $\perp \mathcal{F}$. The latter effectively says "do the impossible" while the former "the impossible was done" with the additional obligation to show how.

Finally, note that our approach takes aspects of both notions of absurdity from (1) and (2) and combines them. In particular, it takes the structural punctuation mark from the structural approach but treats it as a force indicator instead and it takes the absurdity constant from the logical approach but treats it exclusively as the False propositional constant $\mathcal{F}$ instead. Only together do they form the compound $\perp \mathcal{F}$ which we regard as absurdity.

3 Natural deduction rules revisited

In the previous section, we have specified absurdity as the impossible command, i.e., the command to make the False propositional constant true. In this section, let us interpret the natural deduction absurdity rules EFQ, ¬E, ¬I, and RAA with this new conception of absurdity in mind.

First, let us recall that in natural deduction we can generally encounter two basic types of rules:

- logical rules: rules that govern logical connectives (most often in the form of introduction and elimination rules)
- structural rules: rules that govern structural aspects of derivations irrespective of logical connectives involved (typically rules for making or manipulating assumptions)

Note that force indicators should also be regarded as structural aspects since they carry additional information about propositions that goes beyond their logical structure. In other words, just by looking at the logical structure of a proposition alone, we cannot determine what kind of force indicator is associated with it and vice versa.

In the logical approach to absurdity (1), all absurdity rules EFQ, ¬E, ¬I, and RAA are categorized as logical rules. In the structural approach to absurdity (2), ¬E, ¬I, and RAA remain logical rules, since they still explicitly refer to at least one logical connective, however, EFQ becomes a structural rule as it no longer governs any logical connective. As we will see, in our hybrid approach, all absurdity rules EFQ, $\neg$E, $\neg$I, and RAA will actually become mixed rules, i.e., rules that combine features of both logical and structural rules. Let us call this type of rule as semi-structural.

- semi-structural rules: rules that govern structural aspects of derivations but with respect to logical connectives involved

Furthermore, since we are interested only in absurdity, we will simplify the considered natural deduction framework by restricting commands only to the False propositional constant $\mathcal{F}$, i.e., we can only form the impossible command $\perp \mathcal{F}$ (or its hypothetical variant) while general commands such as, e.g., $\perp A$ or $\perp A \wedge B$ will not be allowed. Thus, we will not consider, e.g., rules such as:

$$\frac{\perp A \qquad \perp B}{\perp A \wedge B}$$

for composing commands, etc. In the upcoming presentation, we will sometimes also overload the symbol $\perp$: it will mean absurdity when discussing the standard interpretation of absurdity rules, but imperative force indicator when discussing our new forms of the rules. No confusion should arise, however, as it will always be clear from the context which interpretation is meant.

Let us start with the EFQ rule as it is the most tricky one.

3.1 Ex falso quodlibet

The standard EFQ rule with explicit force indicators (we are assuming here the logical approach (1)):

$$\frac{\vdash \perp}{\vdash A}\ \text{EFQ}$$

can be interpreted as follows: if we assert that absurdity is true, then we can assert that anything is true.

The first attempt at what the EFQ rule might look like in our approach is the following:

$$\frac{\perp \mathcal{F}}{\vdash A}$$

which can be interpreted as: if we are commanded to prove that the False propositional constant is true, then we can assert that anything is true.

Note, however, that in its current form, the rule does not allow us to derive absurdity from itself,[18] which the standard EFQ allows, i.e., from $\vdash \perp$ we can derive $\vdash \perp$ again via EFQ (since the same forces appear in the premise and the conclusion). With the variant of EFQ displayed above this is not possible. If

[18]In other words, the rule is not locally complete – it is too weak as an elimination rule (we return to this topic in Section 3). This could, however, also be considered as a positive as it can, e.g., simplify normalization proofs. We thank Nils Kürbis for this note.

we suppose that $A = \mathcal{F}$, then we could derive with this rule from $\perp \mathcal{F}$ only the assertion $\vdash \mathcal{F}$, but never back the command $\perp \mathcal{F}$ itself. To allow derivations of absurdity from absurdity we present the rule as follows:

$$\frac{\perp \mathcal{F}}{\vdash/\perp A}\ \text{EFQ}^*$$

The schematic notation $\vdash/\perp$ is meant to signify that the resulting force of the conclusion is dependent on whether A is $\mathcal{F}$ or not: If so, it gets the force indicator $\perp$, otherwise $\vdash$. With this rule, we can then easily derive the impossible command from itself, see the following example:

$$\cfrac{\cfrac{\vdots}{\perp \mathcal{F}}}{\cfrac{\perp \mathcal{F}}{\vdash A}\ \text{EFQ}^*}\ \text{EFQ}^*$$

Also, note that EFQ* is no longer a purely logical rule telling us that from $\mathcal{F}$ we can derive A as it also governs structural aspects. Namely, it governs the change of force indicators from the premise to the conclusion. Thus, it is a semi-structural rule.[19]

Now, as we argued above, $\perp \mathcal{F}$ stops the derivation or "ties it into a knot" (as Tennant would say) since it gives us the command and thus also the obligation to make $\mathcal{F}$ true but making $\mathcal{F}$ true is impossible. And thus a stopping point is reached. But if so, how should we then understand a rule whose premise tells us to stop the derivation and yet still provides a way forward in the derivation in the form of a conclusion?

We believe we can make the best sense of this rule if we think about it as a command disobeying rule. After all, any command can be disobeyed and $\perp \mathcal{F}$ is no exception to this. As Dummett put it:

> Likewise the sense of a command is determined by knowing what constitutes obedience to it and what disobedience; (Dummett (1959), p. 8)

See also Dummett (1973), p. 303. What constitutes obedience to the command to stop the derivation seems straightforward, just as what constitutes disobedience to it.

What possible reasons there might be for disobeying and thus avoiding the command $\perp \mathcal{F}$? It is, of course, the very fact that the task itself is impossible, and thus there should be no obligations towards it. In other words, we can think of the EFQ rule as a logical variant of the legal principle of *Impossibilium nulla obligatio est* ("There is no obligation of the impossible") or the Kantian ethical principle "ought implies can" as considered by Martin-Löf (2015): since by $\perp \mathcal{F}$

[19] Note that if we were to treat absurdity as the impossible assertion, i.e., $\vdash \mathcal{F}$, instead of the impossible command $\perp \mathcal{F}$, the EFQ rule would no longer be a semi-structural rule but purely a logical one as it would not govern the structural aspects, namely the change of force indicators.

we are given an impossible task, we seem to be fully justified in not carrying it out, since it cannot be carried out.[20]

Thus, we can view EFQ as a rule stating: if you are given the impossible command that effectively stops the derivation, you can disobey it and continue the derivation either by proving any proposition straightway (i.e., derive $\vdash A$) or by repeating the initial command (i.e., derive $\bot\,\mathcal{F}$ again). Either way, the derivation continues. In other words, the EFQ as a "command disobeying rule" captures the idea that to disobey the impossible command, which has the effect of stopping the derivation, all we need to do is simply *not stop* the derivation, i.e., continue the derivation further with either derivation of an arbitrary assertion $\vdash A$ (i.e., it is does not matter what particular assertion it is, what matters is only the fact that its derivation alone continues the derivation) or by repeating the impossible command $\bot\,\mathcal{F}$.

Remark. Viewing EFQ through an imperative perspective has support not only in standard logical practices (when we reach absurdity we ought to stop and start looking for errors) but also in type-theoretical/programming systems. For example, the $\mathbb{N}_0$-elimination rule of constructive type theory (Martin-Löf (1984)), which corresponds to EFQ, contains in its conclusion a function R_0. This function has a "[vacuous] set of instructions for executing" and, as Martin-Löf himself notes, "it is similar to the programming statement *abort* introduced by Dijkstra." (Martin-Löf (1984), p. 36) What is the semantics of *abort*? According to Dijkstra, it is a mechanism that:

> . . . cannot even "do nothing" in the sense of "leaving things as they are"; it really cannot do a thing. [...] When evoked, the mechanism named "abort" will therefore fail to reach a final state: its attempted activation is interpreted as a symptom of failure. (Dijkstra (1976), p. 26)

Our impossible command $\bot\,\mathcal{F}$ to "prove the unprovable", which results in stopping a derivation, can be seen as corresponding to Dijkstra's "do nothing" (or rather, "don't do a thing") command, which results in aborting a program. The choice of explicit *abort* function seems to be a roundabout way of expressing the idea that there is an imperative force at play.

3.2 Negation elimination

The standard $\neg$E rule (with displayed force indicators):

$$\frac{\vdash \neg A \qquad \vdash A}{\vdash \bot}\ \neg\mathrm{E}$$

can be interpreted as follows: if we assert that both A and $\neg A$ are true, then absurdity follows. And, if we allow the definition $\neg A =_{df} A \to \bot$, the rule becomes just a special case of the implication elimination rule:

[20] See Martin-Löf (2020) who also suggests an interpretation of "$\bot$ *true*" (= "$\vdash \mathcal{F}$" in our notation) in terms of an impossible obligation. In other words, Martin-Löf (2020) considers that obligations already arise from assertions (an idea, as he notes, originating from Peirce (1935), §546). Here, however, we will not go as far, and consider obligations to be arising only from commands. See also, e.g., Martin-Löf (2015).

$$\frac{\vdash A \to \bot \qquad \vdash A}{\vdash \bot} \to\mathrm{E}$$

In our approach, the rule looks as follows:

$$\frac{\vdash \neg A \qquad \vdash A}{\bot\, \mathcal{F}} \neg\mathrm{E}^*$$

We can read it as: if we assert that both A and $\neg A$ is true, then stop the derivation (see above why the impossible command can be naturally understood in these terms).[21] In other words, if A is true and $\neg A$ is true, then applying $\neg$E* leads to the obligation to make $\mathcal{F}$ true. Note that the force of the conclusion is different from the force of the premises, so this should also be regarded as a semi-structural rule.

Observe that we can still view $\neg$E* as a special case of the $\to$E rule, i.e.,

$$\frac{\vdash A \to \mathcal{F} \qquad \vdash A}{\bot\, \mathcal{F}} \neg\mathrm{E}^*$$

the only difference is that it is special not only because the False propositional constant $\mathcal{F}$ appears in the consequent of the implication premise but also because the force of the conclusion changes from assertoric $\vdash$ to imperative $\bot$. The same will hold for $\neg$I* rule as well, which we will discuss below.

3.3 Negation introduction

The standard $\neg$I rule (with added force indicators):

$$\begin{array}{c} [A] \\ \dfrac{\vdash \bot}{\vdash \neg A} \neg\mathrm{I} \end{array}$$

can be interpreted as follows: if assuming that A is true entails absurdity, then we can assert that $\neg A$ is true (and withdraw the assumption A). And, again, if we allow the definition $\neg A =_{df} A \to \bot$, this rule becomes a special case of the implication introduction rule:

$$\begin{array}{c} [A] \\ \dfrac{\vdash \bot}{\vdash A \to \bot} \to\mathrm{I} \end{array}$$

Now, interpreting this rule gets a little tricky. First, note that the premise of this rule is not a straightforward assertion claiming that something is the case since it also depends on A. In other words, $\bot$ in the premise is "not asserted *outright*, but only *conditionally*, under the assumption that A is true" (see Sundholm (2006), p. 603). What is then the status of $\bot$? It is a consequent part of a *hypothetical assertion* that tells us that $\bot$ is true under the assumption that

[21] It is worth noting that in the terminology of Vranas (2010), this rule could be used to make "*mixed* imperative arguments" which have some (or only) assertions as premises and command as the conclusion. See Vranas (2010), p. 60.

A is also true (let us not confuse the hypothetical assertion $\begin{matrix} A \\ \vdash \bot \end{matrix}$ with the categorical assertion $A \to \bot$).

Now, in our approach, the rule takes the following form:

$$\dfrac{\begin{matrix}[A] \\ \bot\,\mathcal{F}\end{matrix}}{\vdash \neg A}\ {\scriptstyle \neg\mathrm{I}^*}$$

Analogously to the premise of the ¬I rule, which is a hypothetical assertion, let us treat the premise of the ¬I* rule as a hypothetical command $\begin{matrix} A \\ \bot\,\mathcal{F} \end{matrix}$, i.e., a command $\mathcal{F}$ depending on the assumption A.[22] This also means that we can then understand $\bot\,\mathcal{F}$ as a categorical command, i.e., a command depending on no assumptions. Since we are dealing here with a hypothetical impossible command depending on the truth of A, we can not only disobey it (for the same reasons as with EFQ) but we can also transform it into a categorical assertion of its negation, i.e., $\vdash \neg A$, and discharge the original assumption that led to it.

Now, the intended interpretation of this rule is as follows: if assuming that A is true entails stopping the derivation (= command to prove the False propositional constant), then we can assert that $\neg A$ is true (and withdraw the assumption A). Note, however, that the standard reading of this rule, i.e., "assuming A, we derive absurdity" can still be applied. What changes is the notion of absurdity: it is not just $\mathcal{F}$, i.e., the False propositional constant, but the impossible command $\bot\,\mathcal{F}$. Finally, note that ¬I* is again a semi-structural rule: it governs both logical connectives and structural aspects, particularly the change of force indicators.

3.4 Reductio ad absurdum

So far, all our considerations have been limited to intuitionistic logic, however, they can be naturally extended toward classical logic as well. All we need to do is add the RAA rule and show that it can be interpreted in a way that is consistent with our interpretation of the rules EFQ*, ¬I*, and ¬E*.

The standard RAA rule (with explicit force indicators):

$$\dfrac{\begin{matrix}[\neg A] \\ \vdash \bot\end{matrix}}{\vdash A}\ {\scriptstyle \mathrm{RAA}}$$

can be interpreted as follows: if assuming that $\neg A$ is true entails absurdity, then we can assert that $\neg A$ is true (and withdraw the assumption $\neg A$).

[22] Hypothetical commands are a topic that would deserve further study. For example, Dummett (1973) concludes that in our society conditional imperatives do not appear with a concession that in theory, they could be possible in "a society in which the giving of commands was a strictly formalized affair." (Dummett (1973), p. 342) We believe that the "society of natural deduction users" could fit the bill, but in this paper, we will not pursue this topic further. The same goes also for the relation between hypothetical commands and literature on conditional obligations (see, e.g., Muñoz and Pummer (2021)).

Note again that this rule is very similar to the ¬I rule ("intuitionistic reductio"), only the propositions in the assumption and conclusion are switched.[23]

In our approach, the rule will look as follows:

$$\dfrac{\begin{array}{c}[\neg A]\\ \bot\,\mathcal{F}\end{array}}{\vdash A}\ \text{RAA}^*$$

The interpretation of RAA* proceeds analogously to the interpretation of ¬I*: if assuming that $\neg A$ is true entails stopping the derivation (= command to prove the False propositional constant), then we can assert that A is true (and withdraw the assumption $\neg A$).

Note that ¬I* is again a semi-structural rule: it governs both logical connectives and structural aspects. Also, recall that structural approach (2) makes RAA into a pure rule. In our hybrid approach, RAA* remains an impure rule since it makes use of two different logical constants: ¬ and $\mathcal{F}$.[24] For an overview of all the rules of natural deduction extended with the impossible command for intuitionistic and classical logic, see Figure 1 (for simplicity, we present only the propositional fragment and assume negation as primitive).

To conclude the presentation of the new absurdity rules, let us consider the following proof of De Morgan's law $\neg(A \vee B) \to (\neg A \wedge \neg B)$ to show their use in practice (namely, ¬I* and ¬E*):

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}[\neg(A\vee B)]^1\\ \vdash \neg(A \vee B)\end{array} \quad \dfrac{\begin{array}{c}[A]^2\\ \vdash A\end{array}}{\vdash A \vee B}\,\vee\mathrm{I}_l}{\bot\,\mathcal{F}}\,\neg\mathrm{E}^*}{\vdash \neg A}\,\neg\mathrm{I}^*_2 \quad \dfrac{\dfrac{\begin{array}{c}[\neg(A\vee B)]^1\\ \vdash \neg(A \vee B)\end{array} \quad \dfrac{\begin{array}{c}[B]^3\\ \vdash B\end{array}}{\vdash A \vee B}\,\vee\mathrm{I}_r}{\bot\,\mathcal{F}}\,\neg\mathrm{E}^*}{\vdash \neg B}\,\neg\mathrm{I}^*_3}{\vdash \neg A \wedge \neg B}\,\wedge\mathrm{I}}{\vdash \neg(A \vee B) \to (\neg A \wedge \neg B)}\,\to\mathrm{I}_1$$

Note that this derivation stops on two separate occasions, specifically at both instances of the negation elimination rule ¬E* that result in deriving the impossible command $\bot\,\mathcal{F}$. Since this command cannot be fulfilled, and thus we have no obligation to obey it, in both cases we can avoid it by discharging the corresponding assumptions and deriving assertions of their negations instead.

Finally, note that while our presentation of the rules for intuitionistic and classical logic is nonstandard due to the inclusion of the new force indicator for commands ⊥ (recall that the indicator ⊢ can be considered to be already there, just left implicit), if we suppress the force indicators, as is typically done, we obtain back the standard presentation of the rules. The only difference is that

[23] This difference turns out to be very important as it allows us to derive $\neg\neg A \to A$ (which is not derivable in intuitionistic logic), and consequently, the double negation biconditional $\neg\neg A \leftrightarrow A$ characteristic for classical logic.

[24] It is worth noting, however, that we can turn RAA* into a pure rule by adopting a natural deduction with higher-level rules (Schroeder-Heister (1984), Schroeder-Heister (2014)).

Structural rule:

$$\begin{array}{c} A \\ \vdash A \end{array} \ \text{Assumption rule}$$

Logical rules:

$$\frac{\vdash A \qquad \vdash B}{\vdash A \wedge B}\,\wedge\mathrm{I} \qquad \frac{\vdash A \wedge B}{\vdash A}\,\wedge\mathrm{E}_l \qquad \frac{\vdash A \wedge B}{\vdash B}\,\wedge\mathrm{E}_r$$

$$\begin{array}{c} [A]^i \\ \dfrac{\vdash B}{\vdash A \to B} \end{array}\,\to\mathrm{I}_i \qquad \frac{\vdash A \to B \qquad \vdash A}{\vdash B}\,\to\mathrm{E}$$

$$\frac{\vdash A}{\vdash A \vee B}\,\vee\mathrm{I}_l \qquad \frac{\vdash B}{\vdash A \vee B}\,\vee\mathrm{I}_r \qquad \frac{\vdash A \vee B \qquad \begin{array}{c}[A]^i \\ \vdash D\end{array} \qquad \begin{array}{c}[B]^j \\ \vdash D\end{array}}{\vdash D}\,\vee\mathrm{E}_{i,j}$$

Semi-structural rules:

$$\begin{array}{c} [A] \\ \dfrac{\bot\, \mathcal{F}}{\vdash \neg A} \end{array}\,\neg\mathrm{I}^* \qquad \frac{\vdash \neg A \qquad \vdash A}{\bot\, \mathcal{F}}\,\neg\mathrm{E}^* \qquad \frac{\bot\, \mathcal{F}}{\vdash/\bot\, A}\,\mathrm{EFQ}^* \qquad \begin{array}{c} [\neg A] \\ \dfrac{\bot\, \mathcal{F}}{\vdash A} \end{array}\,\mathrm{RAA}^*$$

Figure 1: Rules for natural deduction extended with the impossible command for intuitionistic and classical propositional logic.

we denote falsity by $\mathcal{F}$ instead of the more traditional $\bot$. For example, in the case of $\neg\mathrm{I}^*$ this process of removing force indicators will proceed as follows:

$$\begin{array}{c} [A] \\ \dfrac{\bot\, \mathcal{F}}{\vdash \neg A} \end{array}\,\neg\mathrm{I}^* \quad \rightsquigarrow \quad \begin{array}{c} [A] \\ \dfrac{\mathcal{F}}{\neg A} \end{array}\,\neg\mathrm{I}$$

and results in a notational variant of the standard $\neg$I rule. The other direction holds as well, i.e., we can regain the variants with explicit force indicators with a simple syntactical transformation: if the proposition $\mathcal{F}$ occurs on its own (i.e., not as a constituent of another proposition) and not as an assumption, we just add in front of it the imperative force indicator $\bot$, and we add the assertion force indicator $\vdash$ in front of all other propositions (again, with the exception of assumptions). In the case of $\neg$I this will proceed as follows:

$$\begin{array}{c} [A] \\ \dfrac{\mathcal{F}}{\neg A} \end{array}\,\neg\mathrm{I} \quad \rightsquigarrow \quad \begin{array}{c} [A] \\ \dfrac{\bot\, \mathcal{F}}{\vdash \neg A} \end{array}\,\neg\mathrm{I}^*$$

To put it differently, on the propositional level, everything stays the same with the difference that $\mathcal{F}$ will not be considered as absurdity but just as falsity. Our approach affects only the level of speech acts (assertions, commands, etc.).

This also means that our approach has no direct effect on the standard meta-theoretic properties of the underlying propositional systems such as normalization theorem, subformula property, consistency, etc. This is for the simple reason that all these properties are typically established for the propositional level only and the level of speech acts does not come into play in any significant way.

This, however, doesn't mean that no meta-theoretic properties can be established for the speech act level, just that the old propositional ones are not affected. This would be a topic for a separate paper, however, we will at least give some examples. We could, for instance, show that our rules are harmonious, i.e., that they are both locally sound and complete (see Pfenning and Davies (2001), also Jacinto and Read (2017)). To demonstrate local soundness we need to provide a (local) reduction procedure that will show us how to remove detours formed by applying elimination rules immediately after the corresponding introduction rules. This will then demonstrate that the elimination rules are not too strong in the sense that they do not allow us to derive more than the corresponding introduction rules license.

For example, for the rules $\neg\mathrm{I}^*$, $\neg\mathrm{E}^*$ the reduction we obtain would look as follows:

$$\dfrac{\dfrac{\begin{matrix}[A]^n\\ \mathcal{D}\\ \bot\,\mathcal{F}\end{matrix}}{\vdash \neg A}\,{\scriptstyle\neg\mathrm{I}^*_n} \qquad \begin{matrix}\mathcal{D}'\\ \vdash A\end{matrix}}{\bot\,\mathcal{F}}\,{\scriptstyle\neg\mathrm{E}^*} \qquad \Rightarrow_{red} \qquad \begin{matrix}\mathcal{D}'\\ \vdash A\\ \mathcal{D}\\ \bot\,\mathcal{F}\end{matrix}$$

The fully generalized variant (where $\neg\mathrm{I}^*$, $\neg\mathrm{E}^*$ regarded as special cases of $\to$I of $\to$E, i.e., with negation $\neg A$ defined as $A \to \mathcal{F}$) would then look as follows:

$$\dfrac{\dfrac{\begin{matrix}[A]^n\\ \mathcal{D}\\ \vdash/\bot\, B\end{matrix}}{\vdash A \to B}\,{\scriptstyle\to\mathrm{I}^n} \qquad \begin{matrix}\mathcal{D}'\\ \vdash A\end{matrix}}{\vdash/\bot\, B}\,{\scriptstyle\to\mathrm{E}} \qquad \Rightarrow_{red} \qquad \begin{matrix}\mathcal{D}'\\ \vdash A\\ \mathcal{D}\\ \vdash/\bot\, B\end{matrix}$$

Recall that the notation $\vdash/\bot$ is meant to signify that the force is dependent on whether B is $\mathcal{F}$ or not (if so, it gets the $\bot$ force indicator, otherwise $\vdash$).

To demonstrate local completeness we need to provide a (local) expansion procedure that will show us how to re-derive a proposition from itself (with no other assumptions) by applying introduction rules immediately after the corresponding elimination rules. This will then demonstrate that the elimination rules are not too weak in the sense that they do not allow us to derive less than the corresponding introduction rules license.

For $\neg\mathrm{I}^*$ and $\neg\mathrm{E}^*$, this expansion would look as follows:

$$\begin{matrix}\mathcal{D}\\ \vdash \neg A\end{matrix} \qquad \Rightarrow_{exp} \qquad \dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \vdash \neg A\end{matrix} \qquad \begin{matrix}[A]^n\\ \vdash A\end{matrix}}{\bot\,\mathcal{F}}\,{\scriptstyle\neg\mathrm{E}^*}}{\vdash \neg A}\,{\scriptstyle\neg\mathrm{I}^*_n}$$

We omit the generalized variant which can be easily reconstructed.

4 Concluding remarks

We have presented a new approach to absurdity in the context of natural deduction for intuitionistic and classical logic. It is a hybrid approach that combines some of the desirable aspects of both the logical approach (namely, the ability to define negation with the help of absurdity) and the structural approach (namely, the ability to regard EFQ as a kind of structural rule). However, not all desirable aspects of the structural approach can be easily implemented, e.g., even in the hybrid approach, RAA has to be regarded as an impure rule.[25] We can quickly summarize these differences between logical, structural, and hybrid approaches in the following table:

	1) Is EFQ a structural rule?	2) Is RAA a pure rule?	3) Can ¬ be defined via absurdity?
logical	no	no	yes
structural	yes	yes	no
hybrid	yes*	no	yes*

The asterisk in the column "Is EFQ a structural rule?" refers to the fact that in our approach, EFQ is a semi-structural rule, i.e., a rule that governs not only structural features but also logical ones. The other asterisk in the column "Can ¬ be defined via absurdity?" refers to the fact that in our approach negation is defined via the propositional content of absurdity, i.e., the False propositional constant, not absurdity as such.[26]

Our hybrid approach is built on the idea that absurdity can be treated as the impossible command that effectively stops derivations and the only way to continue the derivation is to either disobey the command or discharge the assumptions that lead to it. To show that this conception of absurdity can stand on its own, we have shown how to incorporate it into a natural deduction system for both intuitionistic and classical logic by reinterpreting its rules dealing with absurdity, i.e., EFQ, ¬E, ¬I, and RAA. The basic idea was to interpret $\perp$ not as absurdity but as an imperative force indicator and supplement additionally the False propositional constant $\mathcal{F}$ as the propositional part of the impossible command. Thus, instead of just $\perp$, commonly understood as a propositional constant, we are dealing with the [*imperative force*] + [*proposition*] compound $\perp\mathcal{F}$. Interestingly, not much has to change in the standard natural deduction rules to accommodate this approach to absurdity.

[25] But, as we have mentioned above, there is a possible solution for this, assuming we adopt natural deduction with higher-level rules.

[26] If we were to interpret these questions more strictly, the answers to 1) and 3) could also be no* (since it is a semi-structural rule, not a pure structural rule) and no* (since it utilizes the propositional content of absurdity, not absurdity as such), respectively. However, the important part is that the hybrid approach represents an approach that cannot be reduced just to the logical or the structural one.

For future work, there are several possible directions: conceptual (e.g., exploring the notion of absurdity as the impossible command in the context of other traditionally related notions such as contradiction, inconsistency, or impossibility), technical (e.g., investigating the meta-theoretic properties of the natural deduction system extended with the impossible command), and historical (e.g., examining the connection of our atypical notion of absurdity associated with imperative force to other related ones, e.g., according to Wittgenstein, "contradiction jams" derivations (Wittgenstein (1975), p. 179) and leaves us "no room for action" (p. 185)).

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